Strictly Competitive Games

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Reading materials - Strictly competitive games

- **Watson,** Chapter 12:
  - 3 pages long! This time I think it is too short.
- **Harrington,** section 7.6 (about 3-4 pages as well).
  - Only one graphical example, similar to the "tennis game" we will see in class.
- **Osborne,** Chapter 11:
  - Posted on the course website (highly recommended).
  - It is only 15 pages long, including examples, figures, experiments, etc.

Strictly competitive games

- Some strategic situations involve players with completely opposite interests/incentives.
- We analyze those situations with Strictly Competitive Games.
- They are a type of simultaneous-move games, as those described so far...
  - but with an additional assumption:

Definition

- A two-player, strictly competitive game is a two-player game with the property that, for every two strategy profiles s and s',

$$u_1(s) \geq u_1(s') \text{ and } u_2(s) \leq u_2(s')$$

Strictly competitive games

Definition

A two-player, strictly competitive game is a two-player game with the property that, for every two strategy profiles s and s',

$$u_1(s) \geq u_1(s') \quad \text{and} \quad u_2(s) \leq u_2(s')$$

- **Intuition:** Hence, players have exactly opposite rankings over the outcomes resulting from the strategy profile s and s'.
- Alternatively: if my payoff increases if we play $s = (s_1, s_2)$ rather than $s' = (s_1', s_2')$, then your payoff must decrease.

- An implication is that, in a strictly competitive game,
  - if $u_1(s) = u_1(s')$,then $u_2(s) = u_2(s')$.

- Alternatively, to check if a game is not strictly competitive, we want to find two strategy profiles (cells), s and s' for which players' preferences are aligned, that is,
  - $u_1(s) > u_1(s')$,then $u_2(s) > u_2(s')$

Example 1 - Matching pennies

Player 2

Player 1		Heads	Tails
	Heads	$1, -1$	$-1, 1$
	Tails	$-1, 1$	$1, -1$

- One example of a strategy profile is $s = (H, H)$, and another is $s' = (T, H)$, where

$$u_1(s) > u_1(s') \quad \text{and} \quad u_2(s) < u_2(s')$$

- Importantly, this is true for **any two** strategy profiles: if one player is improving his payoff, the other player is reducing his.
- In fact, many board games satisfy this condition: if we play in such a way that I end up winning, it must be that my opponent loses, and viceversa.
- *Examples*: tennis, chess, football, etc.

Practice: (it will be part of Homework #5 or 6)

For the following games, determine which of them satisfy the definition of strictly competitive games:

1. Matching Pennies (Anticoordination game),
2. Prisoner's Dilemma,
3. Battle of the Sexes (Coordination game).

Matching Pennies

P_2

		Heads	Tails
P_1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

- Check if this game satisfies the definition of strictly competitive games.
  - Recall that we must check that, for any two strategy profiles s and s',

$$u_1(s) > u_1(s') \text{ and } u_2(s) < u_2(s')$$

Matching Pennies

- Comparing each possible pair of outcomes

1. $u_1(H, H) > u_1(H, T)$, i.e., $1 > -1$
   $u_2(H, H) < u_2(H, T)$, i.e., $-1 > 1$

2. $u_1(H, T) < u_1(T, T)$, i.e., $-1 < -1$
   $u_2(H, T) > u_2(T, T)$, i.e., $1 > -1$

3. $u_1(H, H) = u_1(T, T)$, i.e., $1 = 1$
   $u_2(H, H) = u_2(T, T)$, i.e., $-1 = -1$

4. $u_1(H, T) = u_1(T, H)$, i.e., $-1 = -1$
   $u_2(H, T) = u_2(T, H)$, i.e., $1 = 1$

Prisoner's Dilemma

Player 2

Player 1		Confess	Not Confess
	Confess	-5, -5	0, -15
	Not Confess	-15, 0	-1, -1

- Check if this game satisfies the above definition of strictly competitive games.
  - [*Hint*: What happens when you compare (C, C) and (NC, NC)? Preference alignment].

Battle of the Sexes

		Wife	
		Football	Opera
Husband	Football	$\underline{3}, \underline{1}$	0, 0
	Opera	0, 0	$\underline{1}, \underline{3}$

- Check if this game is strictly competitive game:
  - [*Hint*: What happens when you compare (F, O) and (F, F)? Preference alignment].

Zero-sum games

- An interesting class of strictly competitive games: zero-sum games.
- A zero-sum game is an strictly competitive game in which the payoffs of the two players adds up to zero. That is

$$u_1(s_1, s_2) + u_2(s_2, s_1) = 0 \text{ for any strategy pair } (s_1, s_2)$$

alternatively, $u_1(s_1, s_2) = -u_2(s_2, s_1)$.
- For a general strictly competitive game we were saying that:

"if one strategy profile (s_1, s_2) increases my payoff, then...

...such strategy must reduce your payoff"
- but in a zero-sum game we are impossing an stronger assumption:

"the payoff that I gain, is *exactly* what you lose"

Zero-sum games

- The definition of a zero-sum game was satisfied by the matching pennies game....

Player 2

Player 1		Heads	Tails
	Heads	$1, -1$	$-1, 1$
	Tails	$-1, 1$	$1, -1$

since $u_1(s_1, s_2) + u_2(s_2, s_1) = 1 + (-1) = 0$, for any strategy profile that especifies one strategy for player 1 and one for player 2, (s_1, s_2).

Constant-sum games

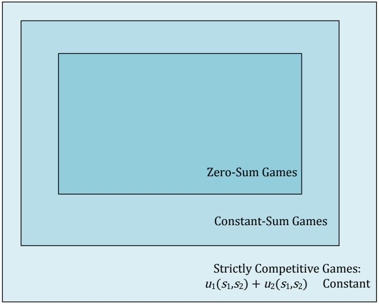
- Some games are not zero-sum games, but they are constant-sum games. (They are of course an special type of strictly competitive game; verify).
- The following *Tennis game* is a constant-sum game

		Player 2	
		Right	Left
Player 1	Right	20, 80	70, 30
	Left	90, 10	30, 70

since $u_1(s_1, s_2) + u_2(s_2, s_1) = 100$, for any strategy profile (s_1, s_2) that especifies one strategy for player 1 and one for player 2.

Constant-sum games

- Hence, for any strategy profile (s_1, s_2)
  - $u_1(s_1, s_2) + u_2(s_1, s_2) = 0$, in zero-sum games
  - $u_1(s_1, s_2) + u_2(s_1, s_2) =$ Constant, in constant-sum games
    - The Constant is exactly equal to zero in zero-sum games
- Therefore



Constant-sum games

- Compact representation of the Tennis game:

Player 2

		Right	Left
Player 1	Right	20	70
	Left	90	30

- We don't need to represent player 2's payoff, since we know that in this constant-sum game

$$u_1(s_1, s_2) + u_2(s_1, s_2) = 100$$

for all strategy profiles (s_1, s_2).
- Hence, player 2's payoffs are 80, 30, 10, and 70

Constant-sum games

- How to solve this class of games?
  - We could use the NE solution concept (implying the need to rely on msNE for most of these games).
  - An alternative, historically developed before John Nash introduced his "NE solution concept," is to use the so-called:
    - **Security strategies**
      (also referred as Max-Min strategies).

Security or Max-min strategy

- Compact representation of the Tennis game:

		Player 2	
		Right	Left
Player 1	Right	20	70
	Left	90	30

- Note that Player 1 wants to maximize his own payoffs, and...
- Player 2 also wants to maximize his own payoffs, which implies minimizing Player 1's payoffs, since we are in a constant-sum game.

Security or Max-min strategy

- Let us put ourselves in the worst case scenario:
  - First, for a given strategy s_1 that player 1 selects, choose the strategy of player 2's that minimizes player 1's payoffs.

$$w_1(s_1) = \min_{s_2} u_1(s_1, s_2)$$

    we refer to $w_1(s_1)$ as the worst payoff that player 1 could achieve by selecting strategy s_1.
  - Alternatively, we can interpret that, if player 1 select strategy s_1, he guarantees to obtain a payoff of at least $w_1(s_1)$.
- A **Security strategy** gives player 1 the best of the worst case scenarios:

$$\max_{s_1} w_1(s_1) = \max_{s_1} \min_{s_2} u_1(s_1, s_2)$$

- The strategy that solves this maximization problem is referred as the Security strategy, or Max-min strategy.

Security or Max-min strategy

$$\max_{s_1} \; w_1(s_1) = \max_{s_1} \min_{s_2} \; u_1(s_1, s_2)$$

- The payoff $\max_{s_1} \; w_1(s_1)$ is usually referred as the Security-payoff level.
- Note what is happening here:
  - I maximize my payoff, given that I know that my opponent will minimize it (because he wants to maximize his own payoff since $u_2(s_1, s_2) = -u_1(s_1, s_2)$).

Security or Max-min strategy

- We can generalize the above definition to mixed strategies, i.e., talking about σ_i rather than s_i.
- Player 1's security payoff level is

$$\max_{\sigma_1} w_1(\sigma_1) = \max_{\sigma_1} \min_{s_2} u_1(\sigma_1, s_2)$$

- And similarly for player 2:

$$\max_{\sigma_2} w_2(\sigma_2) = \max_{\sigma_2} \min_{s_1} u_2(\sigma_2, s_1)$$

Security or Max-min strategy

- Let us first apply Security strategies to the example of the Matching pennies game.
- Afterwards, we will apply the same methodology to the Tennis game.

Security or Max-min strategy

- Note that, in order to find the security (or max-min) strategy for player 1, we need to find

$$\max_{s_1} \min_{s_2} \; u_1(s_1, s_2)$$

- We hence need to first find:
  - $EU_1(p|H)$ conditional on player 2 choosing H.
  - $EU_1(p|T)$ conditional on player 2 choosing T.
- We can then find the min of these two expressions (i.e., their "lower envelope").
- Finally, we can find the max of the min.
- Confused? Ok, let's do one example together. $\longrightarrow$

Security (Max-min) Strategy - Matching Pennies Game

		Player 2 Heads	Tails
Player 1	p Heads	1, -1	-1, 1
	$1 - p$ Tails	-1, 1	1, -1

- 1st step: Find the expected payoff of player 1
  - If player 1 chooses H (In the first column), player 1's EU becomes:

$$EU_1(p|H) = 1 \cdot p + (-1)(1-p) = 2p - 1$$

  - If player 1 chooses T (In the second column), player 1's EU becomes:

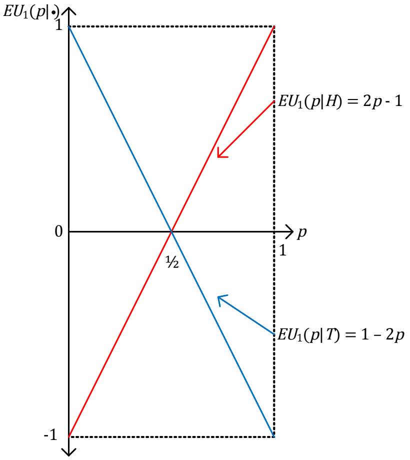
$$EU_1(p|T) = (-1) \cdot p + 1(1-p) = 1 - 2p$$

Security (Max-min) Strategy - Matching Pennies Game

- Remark
  - Note that $EU_1(p|H)$ represents the expected utility that player 1 obtains from randomizing between H (with probability p) and T (with probability $1-p$), conditional on player 2 selecting Heads (in the first column).
  - Do not confuse it with $EU_1(H)$ that we used in msNE, which reflects player 1's expected utility from selecting H with certainty but facing a randomization from his opponent (e.g., player 2 randomizing between H and T with probability q and $1-q$ respectively.

Security (Max-min) Strategy - Matching Pennies Game

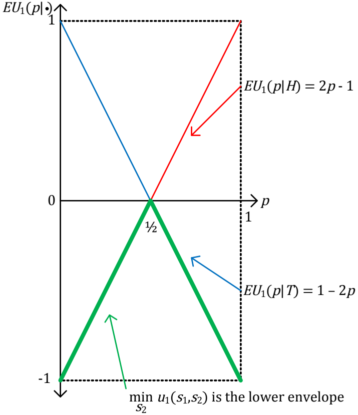
- 2nd step: Let's graphically depict $EU_1(p|H)$ and $EU_1(p|T)$



Security (Max-min) Strategy - Matching Pennies Game

- 3rd step: Identify the lower envelope, i.e.,

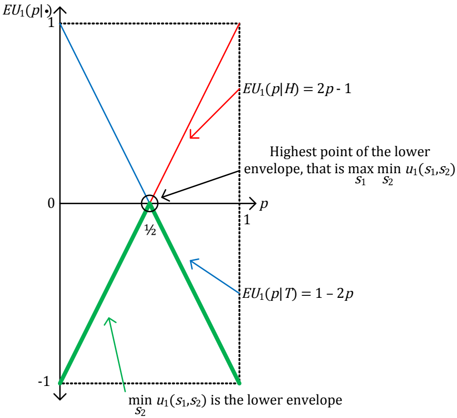
$$\min_{s_2} u_1(s_1, s_2)$$



Security (Max-min) Strategy - Matching Pennies Game

- 4th step: Identify the highest peak of the lower envelope: i.e.,

$$\max_{s_1} \min_{s_2} u_1(s_1, s_2)$$



Security (Max-min) Strategy - Matching Pennies Game

- Summarizing our results...
  - We just found that the Security (or Max-Min) strategy for player 1 is:
    - To choose Heads with probability $p = \frac{1}{2}$.
  - What about player 2?
    - Well, we have to follow the same procedure we used with player 1.
    - Practice on your own (see next two slides).
    - [*Hint*: you should find that player 2 also randomizes with probability $q = \frac{1}{2}$].

Security (Max-min) Strategy - Matching Pennies Game

- Similarly for player 2 (Practice!)

		Player 2	
		q Heads	$1 - q$ Tails
Player 1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

- Expected payoff for player 2:
  - If player 1 plays H (first row):

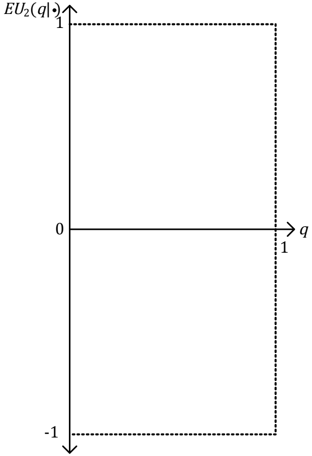
$$EU_2(q|H) =$$

  - If player 2 plays T (second column):

$$EU_2(q|T) =$$

Security (Max-min) Strategy - Matching Pennies Game

- Graphical depiction for player 2



Security (Max-min) Strategy - Tennis Game

- Let's go back to the Tennis game:
  - It is a constant-sum game, since the sum of players' payoffs is equal to a constant (100), for all possible strategy profiles (i.e., for all possible cells in the matrix), but...
  - It is not a zero-sum game, since the sum of players' payoffs is not euqal to zero for all strategy profiles.

		Player 2	
		Right	Left
Player 1	Right	20, 80	70, 30
	Left	90, 10	30, 70

- Let's start with player 1:

Security (Max-min) Strategy - Tennis Game

			Player 2 Right	Left
Player 1	p	Right	20, 80	70, 30
	$1 - p$	Left	90, 10	30, 70

- Player 1's expected payoff:
  - If player 2 chooses Right:

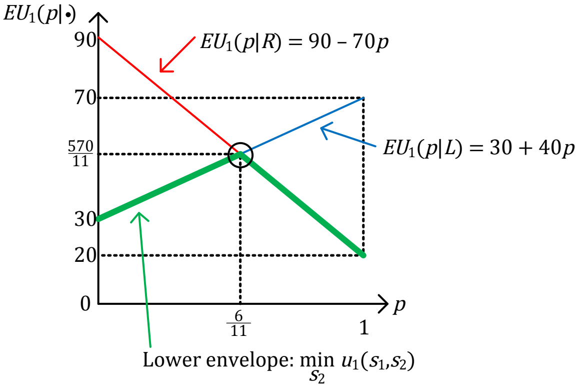
$$EU_1(p|R) = 20p + 90(1-p) = 90 - 70p$$

  - If player 2 chooses Left:

$$EU_1(p|L) = 70p + 30(1-p) = 30 + 40p$$

Security (Max-min) Strategy - Tennis Game

- Graphical depiction for player 1:



Security (Max-min) Strategy - Tennis Game

- **Trick:**

$$\max_{s_1} \min_{s_2} u_1(s_1, s_2)$$

coincides with the value of p for which

$$EU_1(p|R) = EU_1(p|L).$$

That is,

$$70 - 70p = 30 + 40p \implies 60 = 110p \implies p = \frac{6}{11}$$

Hence,

$$EU_1(p|R) = 90 - 70 \cdot \frac{6}{11} = \underbrace{\frac{570}{11}}_{\substack{\text{This is the height of the highest} \\ \text{peak in the lower envelope.}}}$$

Security (Max-min) Strategy - Tennis Game

- Similarly, for player 2

		Player 2: q Right	Player 2: $1 - q$ Left
Player 1	Right	20, 80	70, 30
Player 1	Left	90, 10	30, 70

- Player 2's expected payoff:
  - If player 1 chooses Right (first row):

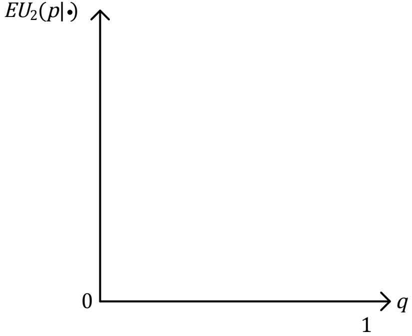
$$EU_2(q|R) = 80q + 30(1 - q) = 30 + 50q$$

  - If player 2 chooses Left (second row):

$$EU_2(q|L) = 10q + 70(1 - q) = 70 - 60q$$

Security (Max-min) Strategy - Tennis Game

- Graphical depiction for player 2 (Practice!):

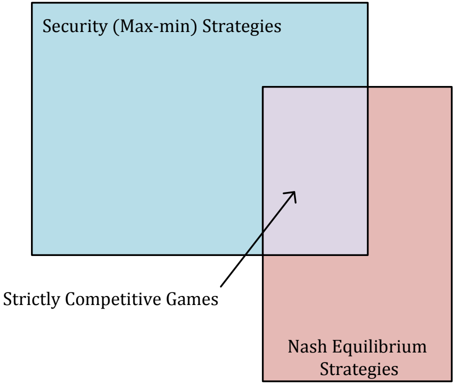


Security or Max-min strategy

- Security strategies were introduced at the beginning of the century before Nash came out with his equilibrium concept...
- For this reason, solving a game using security strategies does not necessarily give us the same equilibrium prediction as if we use Nash equilibrium.
- Although there is one exception! Such exception is, of course, stricty competitive games.

Security or Max-min strategy

- Hence,



- Let us see the relationship between the equilibrium predictions using Security strategies and that using NE.

Security or Max-min strategy

- **Relationship between Security strategies and NE strategies:**
  - If a two-player game is strictly competitive and has a Nash equilibrium $s^* = (s_1^*, s_2^*)$, then...
  - s_1^* is a security strategy for player 1 and s_2^* is a security strategy for player 2.
- That is, if s_1^* is a NE strategy for player 1 in a strictly competitive game, then s_1^* guarantees player 1 at least his security payoff level, regardless of what player 2 does.
- In other words, by playing the NE strategy a player guarantees a payoff equal or higher than that he would obtain by playing the Security (or Maxmin) strategy.
  - You will see that in a very easy exercise in Homework #6.

Practice: (it will be part of Homework #6)

2. Consider the following game:

		Player 2	
		Left	Right
Player 1	Top	6, 0	0, 6
	Bottom	3, 2	6, 0

1. Find every player's maxmin strategy.
2. What is every player's expected payoff from playing her maxmin strategy?
3. Find every player's Nash equilibrium strategy, both using pure strategies (psNE) and using mixed strategies (msNE).
4. What is every player's expected payoff from playing her Nash equilibrium strategy?
5. Compare players' payoff when they play maxmin and Nash equilibrium strategies (from parts (b) and (d), respectively). Which is higher?

What if a game is not strictly competitive?

Consider the following game:

		Player 2	
		X	Y
Player 1	A	$3, 5$	$-1, 1$
	B	$2, 6$	$1, 2$

1. This is indeed an example of a game that **does not satisfy** the definition of strictly competitive games. In particular, we can find two strategy profiles, (A,X) and (A,Y) for which

$$u_1(A, X) > u_1(A, Y) \text{ for player 1,}$$
$$\text{but also } u_2(A, X) > u_2(A, Y) \text{ for player 2!!}$$

What if a game is not strictly competitive?

Player 2

Player 1		X	Y
	A	<u>3</u>, <u>5</u>	-1, 1
	B	2, <u>6</u>	<u>1</u>, 2

1. The game has a unique psNE: (A, X).
2. But, is A the security strategy for player 1?
   - We know that this is the case in strictly competitive games, but...
   - this is not necessarily true in games that are not strictly competitive (such as this one).
   - In order to check if A is a security strategy for player 1, let's find player 1's security strategies $\longrightarrow$

What if a game is not strictly competitive?

			X	Y
Player 1	p	A	<u>3</u>, <u>5</u>	-1, 1
	$1 - p$	B	2, <u>6</u>	<u>1</u>, 2

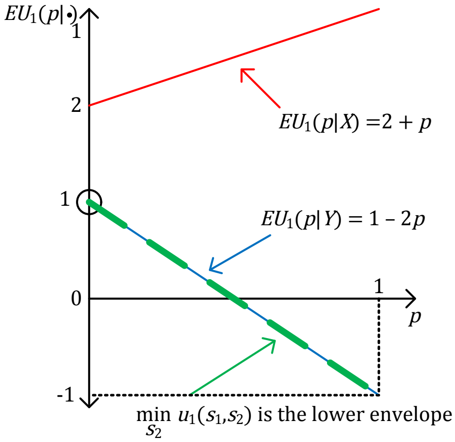
Player 2 (columns X, Y)

1. In order to check if A is the security strategy for player 1, we must find $EU_1(p|X)$ and $EU_1(p|Y)$.

$$
\begin{aligned}
EU_1(p|X) &= 3p + 2(1-p) = 2 + p \\
EU_1(p|Y) &= -1p + 1(1-p) = 1 - 2p
\end{aligned}
$$

What if a game is not strictly competitive?

- Graphical representation for player 1:



What if a game is not strictly competitive?

- $EU_1(p|X)$ and $EU_1(p|Y)$ do not cross for any probability $p \in (0, 1)$.
- $EU_1(p|Y)$ is the minimum of $EU_1(p|X)$ and $EU_1(p|Y)$, i.e., the "Lower envelope."
- The lower envelope is maximized at $p = 0$.
- Hence, player 1 does not assign any probability to action A, but full probability to $B \implies B$ is player 1's security strategy (which differs from his Nash Equilibrium strategy, A).

What if a game is not strictly competitive?

- This confirms our previous result that:
  1. NE and Security strategies **coincide** for strictly competitive games, but...
  2. NE and Security strategies **do not generally coincide** for games that are not strictly competitive.