

Strictly Competitive Games

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Reading materials - Strictly competitive games

- **Osborne**, Chapter 11:
 - Posted on Angel website.
 - It is only 15 pages long, including examples, figures, experiments, etc.

Strictly competitive games

- Some strategic situations involve players with completely opposite interests/incentives.
- We analyze those situations with Strictly Competitive Games.
- They are a type of simultaneous-move games, as those described so far...
 - but with an additional assumption (next slide):

Strictly competitive games

Definition

A two-player, strictly competitive game is a two-player game with the property that, for every two strategy profiles s and s' ,

$$u_1(s) > u_1(s') \quad \text{and} \quad u_2(s) < u_2(s')$$

- **Intuition:** Hence, players have exactly opposite rankings over the outcomes resulting from the strategy profile s and s' .
- Alternatively: if my payoff increases if we play $s = (s_1, s_2)$ rather than $s' = (s'_1, s'_2)$, then your payoff must decrease.

- An implication is that, in a strictly competitive game,
 - if $u_1(s) = u_1(s')$, then $u_2(s) = u_2(s')$.

- Alternatively, to check if a game is not strictly competitive, we want to find two strategy profiles (cells), s and s' for which players' preferences are aligned, that is,
 - $u_1(s) > u_1(s')$, then $u_2(s) > u_2(s')$

Example 1 - Matching pennies

		<i>Player 2</i>	
		Heads	Tails
<i>Player 1</i>	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

- One example of a strategy profile is $s = (H, H)$, and another is $s' = (T, H)$, where

$$u_1(s) > u_1(s') \quad \text{and} \quad u_2(s) < u_2(s')$$

- Importantly, this is true for **any two** strategy profiles: if one player is improving his payoff, the other player is reducing his.
- In fact, many board games satisfy this condition: if we play in such a way that I end up winning, it must be that my opponent loses, and viceversa.
- *Examples:* tennis, chess, football, etc.

Practice:

For the following games, determine which of them satisfy the definition of strictly competitive games:

- 1 Matching Pennies (Anticoordination game),
- 2 Prisoner's Dilemma,
- 3 Battle of the Sexes (Coordination game).

Matching Pennies

		P_2	
		Heads	Tails
P_1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

- Check if this game satisfies the definition of strictly competitive games.
 - Recall that we must check that, for any two strategy profiles s and s' ,

$$u_1(s) > u_1(s') \text{ and } u_2(s) < u_2(s')$$

Matching Pennies

- Comparing each possible pair of outcomes

① $u_1(H, H) > u_1(H, T)$, i.e., $1 > -1$
 $u_2(H, H) < u_2(H, T)$, i.e., $-1 > 1$

② $u_1(H, T) < u_1(T, T)$, i.e., $-1 < -1$
 $u_2(H, T) > u_2(T, T)$, i.e., $1 > -1$

③ $u_1(H, H) = u_1(T, T)$, i.e., $1 = 1$
 $u_2(H, H) = u_2(T, T)$, i.e., $-1 = -1$

④ $u_1(H, T) = u_1(T, H)$, i.e., $-1 = -1$
 $u_2(H, T) = u_2(T, H)$, i.e., $1 = 1$

Prisoner's Dilemma

		<i>Player 2</i>	
		Confess	Not Confess
<i>Player 1</i>	Confess	-5, -5	0, -15
	Not Confess	-15, 0	-1, -1

- Check if this game satisfies the above definition of strictly competitive games.
 - [*Hint*: What happens when you compare (C, C) and (NC, NC) ? Preference alignment].

Battle of the Sexes

		<i>Wife</i>	
		Football	Opera
<i>Husband</i>	Football	<u>3</u> , <u>1</u>	0, 0
	Opera	0, 0	<u>1</u> , <u>3</u>

- Check if this game is strictly competitive game:
 - [*Hint*: What happens when you compare (F, O) and (F, F) ? Preference alignment].

Zero-sum games

- An interesting class of strictly competitive games: zero-sum games.
- A zero-sum game is an strictly competitive game in which the payoffs of the two players adds up to zero. That is

$$u_1(s_1, s_2) + u_2(s_1, s_2) = 0 \text{ for any strategy pair } (s_1, s_2)$$

alternatively, $u_1(s_1, s_2) = -u_2(s_1, s_2)$.

- For a general strictly competitive game we were saying that:

“if one strategy profile (s_1, s_2) increases my payoff, then...

...such strategy must reduce your payoff”

- but in a zero-sum game we are imposing an stronger assumption:

“the payoff that I gain, is *exactly* what you lose”

Zero-sum games

- The definition of a zero-sum game was satisfied by the matching pennies game....

Player 2

		<i>Player 2</i>	
		Heads	Tails
<i>Player 1</i>	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

since $u_1(s_1, s_2) + u_2(s_1, s_2) = 1 + (-1) = 0$, for any strategy profile that specifies one strategy for player 1 and one for player 2, (s_1, s_2) .

Constant-sum games

- Some games are not zero-sum games, but they are constant-sum games. (They are of course an special type of strictly competitive game; verify).
- The following *Tennis game* is a constant-sum game

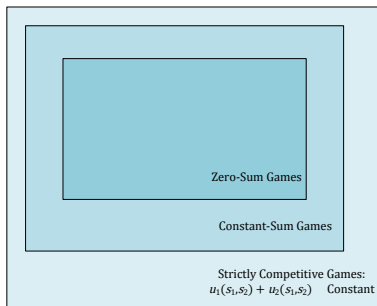
Player 2

		<i>Player 2</i>	
		Right	Left
<i>Player 1</i>	Right	20, 80	70, 30
	Left	90, 10	30, 70

since $u_1(s_1, s_2) + u_2(s_2, s_1) = 100$, for any strategy profile (s_1, s_2) that specifies one strategy for player 1 and one for player 2.

Constant-sum games

- Hence, for any strategy profile (s_1, s_2)
 - $u_1(s_1, s_2) + u_2(s_1, s_2) = 0$, in zero-sum games
 - $u_1(s_1, s_2) + u_2(s_1, s_2) = \text{Constant}$, in constant-sum games
 - The Constant is exactly equal to zero in zero-sum games
- Therefore



Constant-sum games

- Compact representation of the Tennis game:

		<i>Player 2</i>	
		Right	Left
<i>Player 1</i>	Right	20	70
	Left	90	30

- We don't need to represent player 2's payoff, since we know that in this constant-sum game

$$u_1(s_1, s_2) + u_2(s_1, s_2) = 100$$

for all strategy profiles (s_1, s_2) .

- Hence, player 2's payoffs are 80, 30, 10, and 70

Constant-sum games

- How to solve this class of games?
 - We could use the NE solution concept (implying the need to rely on msNE for most of these games).
 - An alternative, historically developed before John Nash introduced his "NE solution concept," is to use the so-called:
 - **Security strategies**
(also referred as Max-Min strategies).

Security or Max-min strategy

- Compact representation of the Tennis game:

		<i>Player 2</i>	
		Right	Left
<i>Player 1</i>	Right	20	70
	Left	90	30

- Note that Player 1 wants to maximize his own payoffs, and...
- Player 2 also wants to maximize his own payoffs, which implies minimizing Player 1's payoffs, since we are in a constant-sum game.

Security or Max-min strategy

- Let us put ourselves in the worst case scenario:
 - First, for a given strategy s_1 that player 1 selects, choose the strategy of player 2's that minimizes player 1's payoffs.

$$w_1(s_1) = \min_{s_2} u_1(s_1, s_2)$$

we refer to $w_1(s_1)$ as the worst payoff that player 1 could achieve by selecting strategy s_1 .

- Alternatively, we can interpret that, if player 1 select strategy s_1 , he guarantees to obtain a payoff of at least $w_1(s_1)$.
- A **Security strategy** gives player 1 the best of the worst case scenarios:

$$\max_{s_1} w_1(s_1) = \max_{s_1} \min_{s_2} u_1(s_1, s_2)$$

- The strategy that solves this maximization problem is referred as the Security strategy, or Max-min strategy.

Security or Max-min strategy

$$\max_{s_1} w_1(s_1) = \max_{s_1} \min_{s_2} u_1(s_1, s_2)$$

- The payoff $\max_{s_1} w_1(s_1)$ is usually referred as the Security-payoff level.
- Note what is happening here:
 - I maximize my payoff, given that I know that my opponent will minimize it (because he wants to maximize his own payoff).

Security or Max-min strategy

- We can generalize the above definition to mixed strategies, i.e., talking about σ_i rather than s_i .
- Player 1's security payoff level is

$$\max_{\sigma_1} w_1(\sigma_1) = \max_{\sigma_1} \min_{s_2} u_1(\sigma_1, s_2)$$

- And similarly for player 2:

$$\max_{\sigma_2} w_2(\sigma_2) = \max_{\sigma_2} \min_{s_1} u_2(\sigma_2, s_1)$$

Security or Max-min strategy

- Let us first apply Security strategies to the example of the Matching pennies game.
- Afterwards, we will apply the same methodology to the Tennis game.

Security or Max-min strategy

- Note that, in order to find the security (or max-min) strategy for player 1, we need to find

$$\max_{s_1} \min_{s_2} u_1(s_1, s_2)$$

- We hence need to first find:
 - $EU_1(p|H)$ conditional on player 2 choosing H.
 - $EU_1(p|T)$ conditional on player 2 choosing T.
- We can then find the min of these two expressions (i.e., their "lower envelope").
- Finally, we can find the max of the min.
- Confused? Ok, let's do one example together. \longrightarrow

Security (Max-min) Strategy - Matching Pennies Game

		<i>Player 2</i>	
		Heads	Tails
<i>Player 1</i>	p Heads	1, -1	-1, 1
	$1 - p$ Tails	-1, 1	1, -1

- 1st step: Find the expected payoff of player 1
 - If player 1 chooses H (In the first column), player 1's EU becomes:

$$EU_1(p|H) = 1 \cdot p + (-1)(1 - p) = 2p - 1$$

- If player 1 chooses T (In the second column), player 1's EU becomes:

$$EU_1(p|T) = (-1) \cdot p + 1(1 - p) = 1 - 2p$$

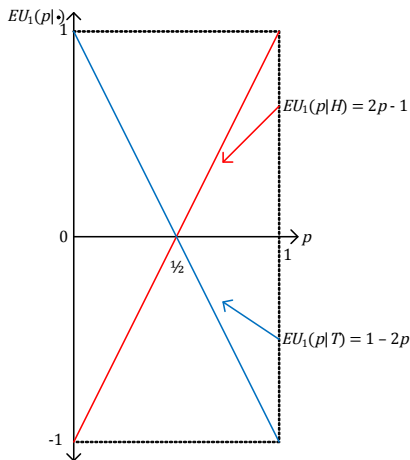
Security (Max-min) Strategy - Matching Pennies Game

- Remark

- Note that $EU_1(p|H)$ represents the expected utility that player 1 obtains from randomizing between H (with probability p) and T (with probability $1 - p$), conditional on player 2 selecting Heads (in the first column).
- Do not confuse it with $EU_1(H)$ that we used in msNE, which reflects player 1's expected utility from selecting H with certainty but facing a randomization from his opponent (e.g., player 2 randomizing between H and T with probability q and $1 - q$ respectively).

Security (Max-min) Strategy - Matching Pennies Game

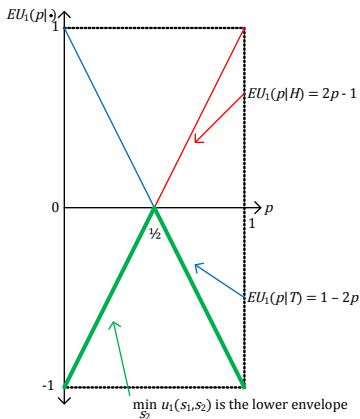
- 2nd step: Let's graphically depict $EU_1(p|H)$ and $EU_1(p|T)$



Security (Max-min) Strategy - Matching Pennies Game

- 3rd step: Identify the lower envelope, i.e.,

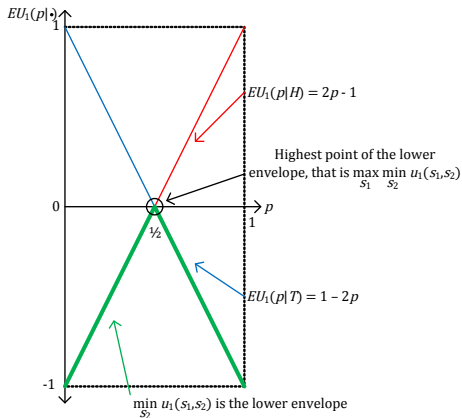
$$\min_{s_2} u_1(s_1, s_2)$$



Security (Max-min) Strategy - Matching Pennies Game

- 4th step: Identify the highest peak of the lower envelope: i.e.,

$$\max_{s_1} \min_{s_2} u_1(s_1, s_2)$$



Security (Max-min) Strategy - Matching Pennies Game

- Summarizing our results...
 - We just found that the Security (or Max-Min) strategy for player 1 is:
 - To choose Heads with probability $p = \frac{1}{2}$.
 - What about player 2?
 - Well, we have to follow the same procedure we used with player 1.
 - Practice on your own (see next two slides).
 - [*Hint*: you should find that player 2 also randomizes with probability $q = \frac{1}{2}$].

Security (Max-min) Strategy - Matching Pennies Game

- Similarly for player 2 (Practice!)

		<i>Player 2</i>	
		q	$1 - q$
<i>Player 1</i>	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

- Expected payoff for player 2:
 - If player 1 plays H (first row):

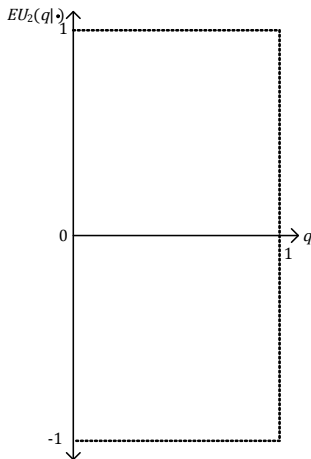
$$EU_2(q|H) =$$

- If player 2 plays T (second column):

$$EU_2(q|T) =$$

Security (Max-min) Strategy - Matching Pennies Game

- Graphical depiction for player 2



Security (Max-min) Strategy - Tennis Game

- Let's go back to the Tennis game:
 - It is a constant-sum game, since the sum of players' payoffs is equal to a constant (100), for all possible strategy profiles (i.e., for all possible cells in the matrix), but...
 - It is not a zero-sum game, since the sum of players' payoffs is not equal to zero for all strategy profiles.

		<i>Player 2</i>	
		Right	Left
<i>Player 1</i>	Right	20, 80	70, 30
	Left	90, 10	30, 70

- Let's start with player 1:

Security (Max-min) Strategy - Tennis Game

		<i>Player 2</i>	
		Right	Left
<i>Player 1</i>	p Right	20, 80	70, 30
	$1 - p$ Left	90, 10	30, 70

- Player 1's expected payoff:
 - If player 2 chooses Right:

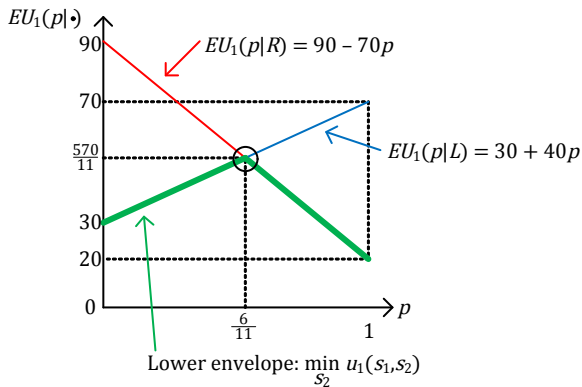
$$EU_1(p|R) = 20p + 90(1 - p) = 90 - 70p$$

- If player 2 chooses Left:

$$EU_1(p|L) = 70p + 30(1 - p) = 30 + 40p$$

Security (Max-min) Strategy - Tennis Game

- Graphical depiction for player 1:



Security (Max-min) Strategy - Tennis Game

- **Trick:**

$$\max_{s_1} \min_{s_2} u_1(s_1, s_2)$$

coincides with the value of p for which

$$EU_1(p|R) = EU_1(p|L).$$

That is,

$$70 - 70p = 30 + 40p \implies 60 = 110p \implies p = \frac{6}{11}$$

Hence,

$$EU_1(p|R) = 90 - 70 \cdot \frac{6}{11} = \underbrace{\frac{570}{11}}$$

This is the height of the highest peak in the lower envelope.

Security (Max-min) Strategy - Tennis Game

- Similarly, for player 2

		<i>Player 2</i>	
		q Right	$1 - q$ Left
<i>Player 1</i>	Right	20, 80	70, 30
	Left	90, 10	30, 70

- Player 2's expected payoff:
 - If player 1 chooses Right (first row):

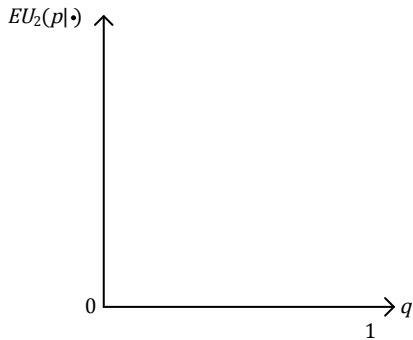
$$EU_2(q|R) = 80q + 30(1 - q) = 30 + 50q$$

- If player 2 chooses Left (second row):

$$EU_2(q|L) = 10q + 70(1 - q) = 70 - 60q$$

Security (Max-min) Strategy - Tennis Game

- Graphical depiction for player 2 (Practice!):

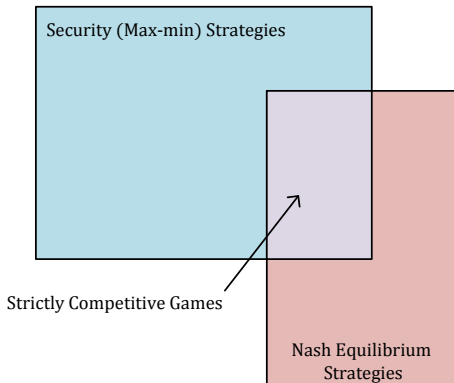


Security or Max-min strategy

- Security strategies were introduced at the beginning of the century before Nash came out with his equilibrium concept...
- For this reason, solving a game using security strategies does not necessarily give us the same equilibrium prediction as if we use Nash equilibrium.
- Although there is one exception!
 - Such exception is, of course, strictly competitive games.

Security or Max-min strategy

- Hence,



- Let us see the relationship between the equilibrium predictions using Security strategies and that using NE.

Security or Max-min strategy

- **Relationship between Security strategies and NE strategies:**
 - If a two-player game is strictly competitive and has a Nash equilibrium $s^* = (s_1^*, s_2^*)$, then...
 - s_1^* is a security strategy for player 1 and s_2^* is a security strategy for player 2.
- That is, if s_1^* is a NE strategy for player 1 in a strictly competitive game, then s_1^* guarantees player 1 at least his security payoff level, regardless of what player 2 does.
- In other words, by playing the NE strategy a player guarantees a payoff equal or higher than that he would obtain by playing the Security (or Maxmin) strategy.

Practice:

2. Consider the following game:

		<i>Player 2</i>	
		Left	Right
<i>Player 1</i>	Top	6, 0	0, 6
	Bottom	3, 2	6, 0

- 1 Find every player's maxmin strategy.
- 2 What is every player's expected payoff from playing her maxmin strategy?
- 3 Find every player's Nash equilibrium strategy, both using pure strategies (psNE) and using mixed strategies (msNE).
- 4 What is every player's expected payoff from playing her Nash equilibrium strategy?
- 5 Compare players' payoff when they play maxmin and Nash equilibrium strategies (from parts (b) and (d), respectively). Which is higher?

What if a game is not strictly competitive?

Consider the following game:

		<i>Player 2</i>	
		X	Y
<i>Player 1</i>	A	3, 5	-1, 1
	B	2, 6	1, 2

- ① This is indeed an example of a game that **does not satisfy** the definition of strictly competitive games. In particular, we can find two strategy profiles, (A,X) and (A,Y) for which

$$u_1(A, X) > u_1(A, Y) \text{ for player 1,}$$

but also $u_2(A, X) > u_2(A, Y)$ for player 2!!

What if a game is not strictly competitive?

		<i>Player 2</i>	
		X	Y
<i>Player 1</i>	A	<u>3</u> , <u>5</u>	-1, 1
	B	2, <u>6</u>	<u>1</u> , 2

- 1 The game has a unique psNE: (A, X) .
- 2 But, is A the security strategy for player 1?
 - We know that this is the case in strictly competitive games, but...
 - this is not necessarily true in games that are not strictly competitive (such as this one).
 - In order to check if A is a security strategy for player 1, let's find player 1's security strategies \longrightarrow

What if a game is not strictly competitive?

		<i>Player 2</i>	
		X	Y
<i>Player 1</i>	p A	3, 5	-1, 1
	$1-p$ B	2, <u>6</u>	<u>1</u> , 2

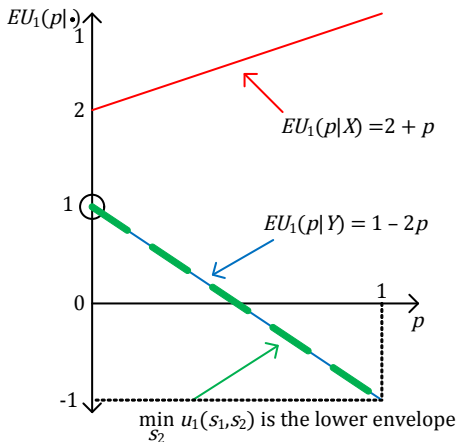
- 1 In order to check if A is the security strategy for player 1, we must find $EU_1(p|X)$ and $EU_1(p|Y)$.

$$EU_1(p|X) = 3p + 2(1-p) = 2 + p$$

$$EU_1(p|Y) = -1p + 1(1-p) = 1 - 2p$$

What if a game is not strictly competitive?

- Graphical representation for player 1:



What if a game is not strictly competitive?

- $EU_1(p|X)$ and $EU_1(p|Y)$ do not cross for any probability $p \in (0, 1)$.
- $EU_1(p|Y)$ is the minimum of $EU_1(p|X)$ and $EU_1(p|Y)$, i.e., the "Lower envelope."
- The lower envelope is maximized at $p = 0$.
- Hence, player 1 does not assign any probability to action A , but full probability to B
 - That is, B is player 1's security strategy (which differs from his Nash Equilibrium strategy, A).

What if a game is not strictly competitive?

- This confirms our previous result that:
 - 1 NE and Security strategies **coincide** for strictly competitive games, but...
 - 2 NE and Security strategies **do not generally coincide** for games that are not strictly competitive.

