Mixed strategy Nash equilibrium

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Looking back...

- So far we have been able to find the NE of a relatively large class of games with complete information:
  - Games with two or several ($n > 2$) players.
  - Games where players select among discrete or continuous actions.
- But, can we assure that all complete information games where players select their actions simultaneously have a NE?
  - We couldn't find a NE for the matching pennies game!! (Next slide)
  - We will be able to claim existence of a NE if we allow players to randomize their actions.

Remembering the "matching pennies" game...

- Recall that this was an example of an anti-coordination game:

		P_2	
		Head	Tail
P_1	Head	$\underline{1}$, -1	-1, $\underline{1}$
	Tail	-1, $\underline{1}$	$\underline{1}$, -1

  Indeed, there is no strategy pair in which players select a particular action 100% of the times.

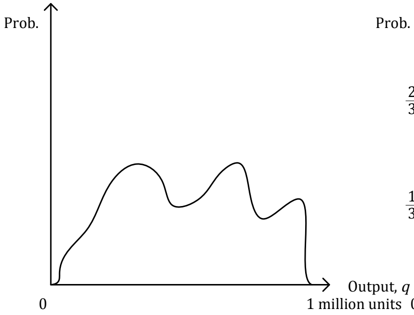
- We need to allow players to randomize their choices.

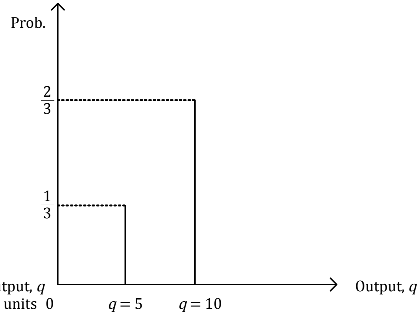
Mixed strategy Nash equilibrium

- Tadelis: Chapter 6.
- First, note that if a player plays more than one strategy with strictly positive probability, then he must be indifferent between the strategies he plays with strictly positive probability.
- **Notation**: "non-degenerate" mixed strategies denotes a set of strategies that a player plays with strictly positive probability.
    - Whereas "degenerate" mixed strategy is just a pure strategy (because of degenerate probability distribution concentrates all its probability weight at a single point).

Degenerate Probability Distributions

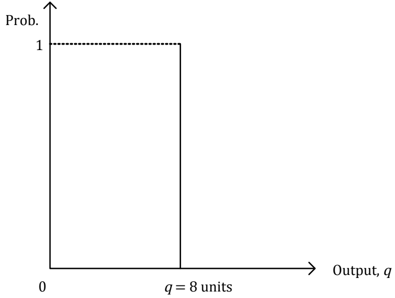
- Example of non-degenerate probability distributions





Degenerate Probability Distributions

- Example of a degenerate probability distribution



- The player (e.g., firm) puts all probability weight (100%) on only one of its possible actions: $q = 8$.

- **Definition of mixed strategy:**
  - Consider player i's finite strategy space $S_i = (s_1, s_2, ..., s_m)$.
  - We can then define $\triangle S_i$ to be the simplex of S_i, i.e., the set of all probability distributions over S_i.
    - (Figures for $m = 2$ and $m = 3$)
  - Therefore, a mixed strategy is an element (i.e., a point) of the simplex, $\sigma_i \in \triangle S_i$

$$\sigma_i = \{\sigma_i(s_1), \sigma_i(s_2), ..., \sigma_i(s_m)\}$$

    where $\sigma_i(s_k)$ denotes the probability that player i plays the pure strategy s_k.
  - As usual, $\sigma_i(s_k) \geq 0$ for all $k = \{1, 2, ..., m\}$, and $\sum_{s_k \in S_i} \sigma_i(s_k) = 1$.

- **Definition of mixed strategy (cont'd):**
  - As usual, we require that:
    - $\sigma_i(s_k) \geq 0$ for all $k = \{1, 2, ..., m\}$, and
    - $\sum_{s_k \in S_i} \sigma_i(s_k) = 1$.
  - When a pure strategy s_k receives a strictly positive probability by σ_i, i.e., $\sigma_i(s_k) > 0$, we say that it is in the support of the mixed strategy σ_i.
  - Otherwise, pure strategy s_k is not in its support.

- **Definition of mixed strategy (cont'd):**
  - What about defining mixed strategies for continuous actions spaces, e.g., $s_i \in \mathbb{R}_+$?
    - We then need to rely on cdf's.
    - A mixed strategy for player i is a cdf

$$F_i : S_i \rightarrow [0, 1]$$

    - where, as usual, for a given value x, $F_i(x)$ represents $F_i(x) = \Pr\{s_i \leq x\}$.
    - If $F_i(x)$ has a density $f_i(x)$, then $f_i(x)$ can be understood as the probability of strategy $s_i = x$ being selected by player i's mixed strategy.

- **Definition of msNE:**
  - Consider a strategy profile $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ where σ_i is a mixed strategy for player i. σ is a msNE if and only if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}) \text{ for all } s_i' \in S_i \text{ and for all } i$$

  - That is, σ_i is a best response of player i to the strategy profile σ_{-i} of the other $N-1$ players, i.e., $\sigma_i = BR_i(\sigma_{-i})$.

- **Remark:**
- Note that we wrote $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\mathbf{s}'_i, \sigma_{-i})$ instead of $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\boldsymbol{\sigma}'_i, \sigma_{-i})$.
- **Why?**
  - If a player was using σ'_i, then he would be indifferent between all pure strategies to which σ'_i puts a positive probability, for example between $\hat{s}_i$ and $\check{s}_i$.
  - That is why it suffices to check that no player has a profitable pure-strategy deviation.

Example 1:Matching pennies

- **Matching pennies**

			Player 2	
			q	$1-q$
			Heads	Tails
Player 1	p	Heads	$1, -1$	$-1, 1$
	$1-p$	Tails	$-1, 1$	$1, -1$

- **Two alternative interpretations of players' randomization:**
  - If player 1 is using a mixed strategy, he must be indifferent between Heads and Tails
  - Alternatively, if player 1 is indifferent between Heads and Tails, it must be that player 2 mixes with a probability q such that player 1 is made indifferent between Heads and Tails:

$$EU_1(H) = EU_1(T) \iff 1q + (1-q)(-1) = (-1)q + 1(1-q)$$

Matching pennies

- **Matching pennies** (example of a normal form game with no psNE):

			Player 2	
			q	$1-q$
			Heads	Tails
Player 1	p	Heads	$1,-1$	$-1,1$
	$1-p$	Tails	$-1,1$	$1,-1$

- Solving for the EU comparison, we obtain

$$EU_1(H) = EU_1(T) \iff 1q + (1-q)(-1) = (-1)q + 1(1-q)$$

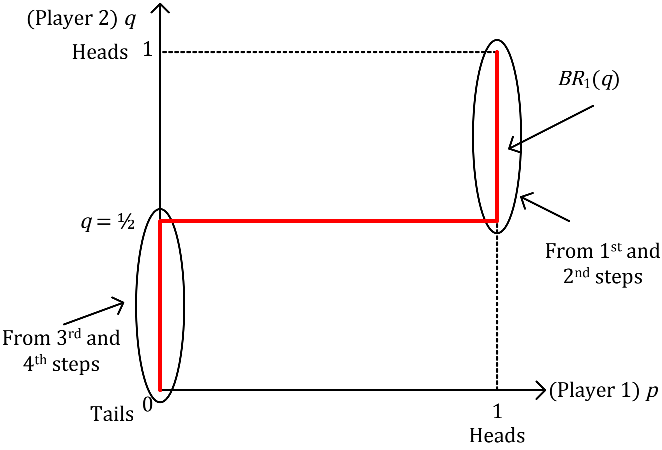
$$q = \frac{1}{2} \longrightarrow \text{Graphical Interpretation}$$

Matching pennies

- How to interpret this cutoff of $q = \frac{1}{2}$ graphically?

    1. We know that if $q > \frac{1}{2}$, then player 2 is very likely playing Heads. Then, player 1 prefers to play Heads as well ($p = 1$).
        - Alternatively, note that $q > \frac{1}{2}$ implies $EU_1(H) > EU_1(T)$.
    2. Go to the figure on the next slide, and draw $p = 1$ for every $q > \frac{1}{2}$.
    3. If $q < \frac{1}{2}$, player 2 is likely playing Tails. Then, player 1 prefers to play Tails as well ($p = 0$).
    4. Graphically, draw $p = 0$ for every $q < \frac{1}{2}$.

Matching pennies



Matching pennies

- Similarly, if player 2 is using a mixed strategy, it must be that he is indifferent between Heads and Tails:

$$EU_2(H) = EU_2(T)$$

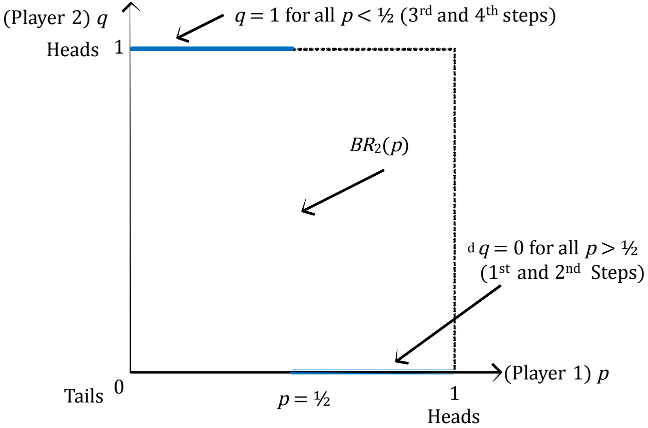
$$(-1)p + 1(1-p) = 1p + (-1)(1-p) \iff p = \frac{1}{2}$$

- (See figure after next slide)

Matching pennies

- Player 2
  1. We know that if $p > \frac{1}{2}$, player 1 is likely playing heads. Then player 2 wants to play tails instead, i.e., $q = 0$.
  2. Go to the figure on the next slide, and draw $q = 0$ for all $p > \frac{1}{2}$.
  3. If $p < \frac{1}{2}$, player 1 is likely playing tails. Then player 2 wants to play heads, i.e., $q = 1$.
  4. Graphically, draw $q = 1$ for all $p < \frac{1}{2}$.

Matching pennies



Matching pennies

- We can represent these BRFs as follows:
  - **Player 1**

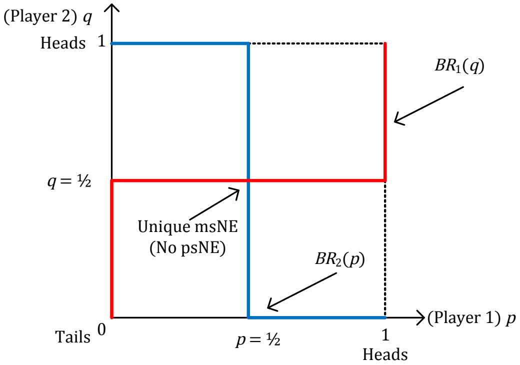
$$BR_1(q) = \begin{cases} \text{Heads if } q > \frac{1}{2} \\ \{\text{Heads, Tails}\} \text{ if } q = \frac{1}{2} \\ \text{Tails if } q < \frac{1}{2} \end{cases}$$

    - Player 1 is indifferent between Heads and Tails when q is exactly $q = \frac{1}{2}$
  - **Player 2**

$$BR_2(p) = \begin{cases} \text{Tails if } p > \frac{1}{2} \\ \{\text{Heads, Tails}\} \text{ if } p = \frac{1}{2} \\ \text{Heads if } p < \frac{1}{2} \end{cases}$$

    - Player 2 is indifferent between Heads and Tails when p is exactly $p = \frac{1}{2}$

Matching pennies



- **Player 1**: When $q > \frac{1}{2}$, Player 1 prefers to play Heads $(p = 1)$; otherwise, Tails.
- **Player 2**: When $p > \frac{1}{2}$, Player 2 prefers to play Tails $(q = 0)$; otherwise, Heads.

Matching pennies

- Therefore, the msNE of this game can be represented as

$$\left\{ \left(\frac{1}{2}H, \frac{1}{2}T \right), \left(\frac{1}{2}H, \frac{1}{2}T \right) \right\}$$

where the first parenthesis refers to player 1(row player), and the player 2(column player).

Battle of the sexes

2. **Battle of the sexes** (example of a normal form game with 2 psNE already!):

			Wife	
			q	$1 - q$
			Football	Opera
Husband	p	Football	$\underline{3}, \underline{1}$	$0, 0$
	$1 - p$	Opera	$0, 0$	$\underline{1}, \underline{3}$

If the Husband is using a mixed strategy, it must be that he indifferent between Football and Opera:

$$\begin{aligned} EU_1(F) &= EU_1(O) \\ 3q + 0(1-q) &= 0q + 1(1-q) \\ 3q &= 1 - q \\ 4q &= 1 \implies q = \frac{1}{4} \end{aligned}$$

Battle of the sexes

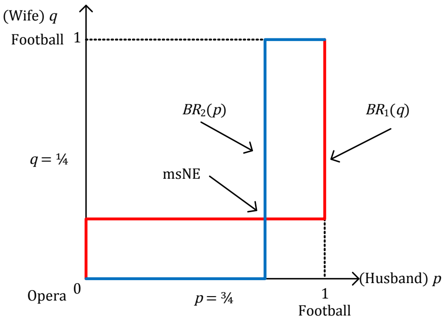
Similarly, if the Wife is using a mixed strategy, it must be that she is indifferent between Football and Opera:

$$\left.\begin{aligned} EU_2(F) &= EU_2(O) \\ & \\ & \\ p &= \frac{3}{4} \end{aligned}\right\} \text{Practice!}$$

Therefore, the msNE of this game can be represented as

$$\text{msNE} = \left\{ \underbrace{\left(\frac{3}{4}F, \frac{1}{4}O\right)}_{\text{Husband}}, \underbrace{\left(\frac{1}{4}F, \frac{3}{4}O\right)}_{\text{Wife}} \right\}$$

Battle of the sexes



- **Husband:** When $q > \frac{1}{4}$, he prefers to go to the Football game $(p = 1)$; otherwise, the Opera.
- **Wife:** When $p > \frac{3}{4}$, she prefers to go to the Football game $(q = 1)$; otherwise, the Opera.

Battle of the sexes

- Best Responses for Battle of the Sexes are hence:
  - **Player 1 (Husband)**

$$BR_1(q) = \begin{cases} \text{Football if } q > \frac{1}{4} \\ \{\text{Football, Opera}\} \text{ if } q = \frac{1}{4} \\ \text{Opera if } q < \frac{1}{4} \end{cases}$$

  - **Player 2 (Wife)**

$$BR_2(p) = \begin{cases} \text{Football if } p > \frac{3}{4} \\ \{\text{Football, Opera}\} \text{ if } p = \frac{3}{4} \\ \text{Opera if } p < \frac{3}{4} \end{cases}$$

Battle of the sexes

- Note the differences in the cutoffs: They reveal each player's preferences.
  - **Husband:** "I will go to the football game as long as there is a slim probability that my wife will be there."
  - **Wife:** "I will only go to the football game if there is more than a 75% chance my husband will be there."

Prisoner's Dilemma

3. **Prisoner's Dilemma** (One psNE, but are there any msNE?):

			Player 2	
			q	$1 - q$
			Confess	Not Confess
Player 1	p	Confess	$\underline{-5}$, $\underline{-5}$	$\underline{0}$, -15
	$1 - p$	Not Confess	-15, $\underline{0}$	-1, -1

If the first player is using a mixed strategy, it must be that he indifferent between Confess and Not Confess:

$$\begin{aligned} EU_1(C) &= EU_1(NC) \\ -5q + 0(1-q) &= -15q + (-1)(1-q) \\ -5q &= -15q - 1 + q \\ 9q &= -1 \implies q = -\frac{1}{9}? \end{aligned}$$

Prisoner's Dilemma

- Similarly, if player 2 is using a mixed strategy, it must be that she is indifferent between Confess and Not Confess:

$$\begin{aligned} EU_2(C) &= EU_2(NC) \\ -5p + 0(1-p) &= -15p + (-1)(1-p) \\ -5p &= -15p - 1 + p \\ 9p &= -1 \implies p = -\frac{1}{9} \end{aligned}$$

- Hence, such msNE would not assign any positive weight to strategies that are strictly dominated.
  - Some textbooks refer to this result by saying that "the support of the msNE is positive only for strategies that are not strictly dominated."

Tennis game (msNE with three available strategies)

4. **Tennis game** (No psNE, but how do we operate with 3 strategies?):

				Player 2	
				q	$1 - q$
			F	C	B
		F	$0, \underline{5}$	$2, 3$	$2, 3$
Player 1	p	C	$2, 3$	$1, \underline{5}$	$\underline{3}, 2$
	$1 - p$	B	$\underline{5}, 0$	$\underline{3}, 2$	$2, \underline{3}$

- Remember this game? We used it as an example of how to delete an strategy that was strictly dominated by the combination of two strategies of that player.
    - Let's do it again.

Tennis game (msNE with three available strategies)

- F is strictly dominated for Player 1:

Player 2

Player 1	F	C (q)	B ($1 - q$)
F	0, 5	2, 3	2, 3
$\frac{1}{3}$C , $\frac{2}{3}$B	4, 1	$\frac{7}{3}$, 3	$\frac{7}{3}$, $\frac{8}{3}$

$\frac{1}{3}(2) + \frac{2}{3}(5) = \frac{12}{3} = 4$

$\frac{1}{3}(3) + \frac{2}{3}(0) = 1$

$\frac{1}{3}(1) + \frac{2}{3}(3) = \frac{7}{3}$

$\frac{1}{3}(5) + \frac{2}{3}(2) = \frac{9}{3} = 3$

$\frac{1}{3}(3) + \frac{2}{3}(2) = \frac{7}{3}$

$\frac{1}{3}(2) + \frac{2}{3}(3) = \frac{8}{3}$

- We can hence rule out F from Player 1 because it is strictly dominated by $(\frac{1}{3}C, \frac{2}{3}B)$.

Tennis game (msNE with three available strategies)

- After deleting F from Player 1's available actions, we are left with:

		Player 2		
		F	C	B
Player 1	C	~~2, 3~~	1, 5	3, 2
	B	~~5, 0~~	3,2	2, 3

- Where we can rule out F from Player 2 because of being strictly dominated by C.

Tennis game (msNE with three available strategies)

- Once strategy F has been deleted for both players, we are left with:

		Player 2: C (q)	Player 2: B ($1-q$)
Player 1	C (p)	1, $\underline{5}$	$\underline{3}$, 2
Player 1	B ($1-p$)	$\underline{3}$, 2	2, $\underline{3}$

- But we cannot identify any psNE, Let's check for msNE:
- If the first player is using a mixed strategy, it must be that he indifferent between C and B:

$$\left.\begin{aligned} EU_1(C) &= EU_1(B) \quad \ldots. \\ q &= \frac{1}{3} \end{aligned}\right\} \text{Practice!}$$

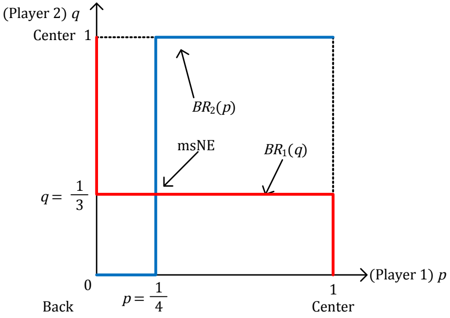
Tennis game (msNE with three available strategies)

- Similarly, if player 2 is using a mixed strategy, it must be that she is indifferent between C and B:

$$
\left.
\begin{aligned}
EU_2(C) &= EU_2(NC) \quad \ldots \\
\\
\\
p &= \frac{1}{4}
\end{aligned}
\right\} \text{Practice!}
$$

- (See figure on next slide)

Tennis game (msNE with three available strategies)



- **Player 1:** If $q > \frac{1}{3}$, then Player 1 prefers Back $(p = 0)$; otherwise Center.
- **Player 2:** If $p > \frac{1}{4}$, then Player 2 prefers Center $(q = 1)$; otherwise Back.

Tennis game (msNE with three available strategies)

- Best Responses in the Tennis Game
  - **Player 1**

$$BR_1(q) = \begin{cases} \text{Back if } q > \frac{1}{4} \\ \{\text{Center, Back}\} \text{ if } q = \frac{1}{4} \\ \text{Center if } q < \frac{1}{4} \end{cases}$$

    - (Recall that $p = 0$ implies playing strategy back with probability one).
  - **Player 2**

$$BR_2(p) = \begin{cases} \text{Center if } p > \frac{1}{4} \\ \{\text{Center, Back}\} \text{ if } p = \frac{1}{4} \\ \text{Back if } p < \frac{1}{4} \end{cases}$$

A few tricks we just learned...

- **Indifference:** If it is optimal to randomize over a collection of pure strategies, then a player receives the same expected payoff from each of those pure strategies.
  - He must be indifferent between those pure strategies over which he randomizes.
- **Odd number:** In almost all finite games (games with a finite set of players and available actions), there is a finite and odd number of equilibria.
  - *Examples*: 1 NE in matching pennies (only one msNE), 3 NE in BoS (two psNE, one msNE), 1 in PD (only one psNE), etc.
- **Never use strictly dominated strategies:** If a pure strategy does not survive the IDSDS, then a NE assigns a zero probability to that pure strategy.
  - *Example*: PD game, where NC is strictly dominated, it does not receive any positive probability.

What if players have three undominated strategies?

- Consider the rock-paper-scissors game

		Player 2		
		Rock	Paper	Scissors
	Rock	0, 0	-1, $\underline{1}$	$\underline{1}$, -1
Player 1	Paper	$\underline{1}$, -1	0, 0	-1, $\underline{1}$
	Scissors	-1, $\underline{1}$	$\underline{1}$, -1	0, 0

- First, note that neither player selects a pure strategy (with 100% probability).

What if players have three undominated strategies?

- Second, every player must be mixing between all his three possible actions, R, P and S.

Player 2

If Player 1 only mixes between Rock and Paper

Player 1		Rock	Paper	Scissors
	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

- **Otherwise:** if P1 mixes only between Rock and Paper, then Player 2 prefers to respond with Paper rather than Rock.
- But if Player 2 never uses Rock, then Player 1 gets a higher payoff with Scissors than Paper. **Contradicton!**
- Then players cannot be mixing between only two of their available strategies.

What if players have three undominated strategies?

- Are you suspecting that the msNE is $\sigma = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$? You're right!

		Player 2		
		Rock	Paper	Scissors
	Rock	0, 0	-1, 1	1, -1
Player 1	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

- We must make every player indifferent between using Rock, Paper, or Scissors.
- That is, $u_1(Rock, \sigma_2) = u_1(Paper, \sigma_2) = u_1(Scissors, \sigma_2)$ for Player 1, and
- $u_2(\sigma_1, Rock) = u_2(\sigma_1, Paper) = u_2(\sigma_1, Scissors)$ for Player 2.

What if players have three undominated strategies?

- Let's separately find each of these expected utilities.
- If player 1 chooses Rock (first row), he obtains

$$
\begin{aligned}
u_1(Rock, \sigma_2) &= 0\sigma_2(R) + (-1)\sigma_2(P) + 1(1 - \sigma_2(R) - \sigma_2(P)) \\
&= -1\sigma_2(P) + 1 - \sigma_2(R) - \sigma_2(P)
\end{aligned}
$$

Player 2

First Row

Player 1		$\sigma_2(R)$ Rock	$\sigma_2(P)$ Paper	$1 - \sigma_2(R) - \sigma_2(P)$ Scissors
	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

What if players have three undominated strategies?

- If player 1 chooses Paper (second row), he obtains

$$\begin{aligned} u_1(Paper, \sigma_2) &= 1\sigma_2(R) + 0\sigma_2(P) + (-1)(1 - \sigma_2(R) - \sigma_2(P)) \\ &= \sigma_2(R) - 1 + \sigma_2(R) + \sigma_2(P) \end{aligned}$$

Player 2

Player 1		$\sigma_2(R)$ Rock	$\sigma_2(P)$ Paper	$1 - \sigma_2(R) - \sigma_2(P)$ Scissors
	Rock	0, 0	-1, 1	1, -1
Second Row	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

What if players have three undominated strategies?

- If player 1 chooses Scissors (third row), he obtains

$$\begin{aligned} u_1(Scissors, \sigma_2) &= (-1)\sigma_2(R) + 1\sigma_2(P) + 0(1 - \sigma_2(R) - \sigma_2(P)) \\ &= -\sigma_2(R) + \sigma_2(P) \end{aligned}$$

Player 2

		$\sigma_2(R)$ Rock	$\sigma_2(P)$ Paper	$1 - \sigma_2(R) - \sigma_2(P)$ Scissors
	Rock	0, 0	-1, 1	1, -1
Player 1	Paper	1, -1	0, 0	-1, 1
Third Row	Scissors	-1, 1	1, -1	0, 0

What if players have three undominated strategies?

- Making the three expected utilities

$$\begin{aligned} u_1(Rock, \sigma_2) &= -1\sigma_2(P) + 1 - \sigma_2(R) - \sigma_2(P), \\ u_1(Paper, \sigma_2) &= \sigma_2(R) - 1 + \sigma_2(R) + \sigma_2(P), \text{ and} \\ u_1(Scissors, \sigma_2) &= -\sigma_2(R) + \sigma_2(P) \end{aligned}$$

equal to each other, we obtain

$$\sigma_2(R) = \sigma_2(P) = 1 - \sigma_2(R) - \sigma_2(P)$$

- Hence, player 2 assigns the same probability weights to his three available actions, thus implying

$$\sigma_2^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

- A similar argument is applicable to player 1, since players' payoffs are symmetric.