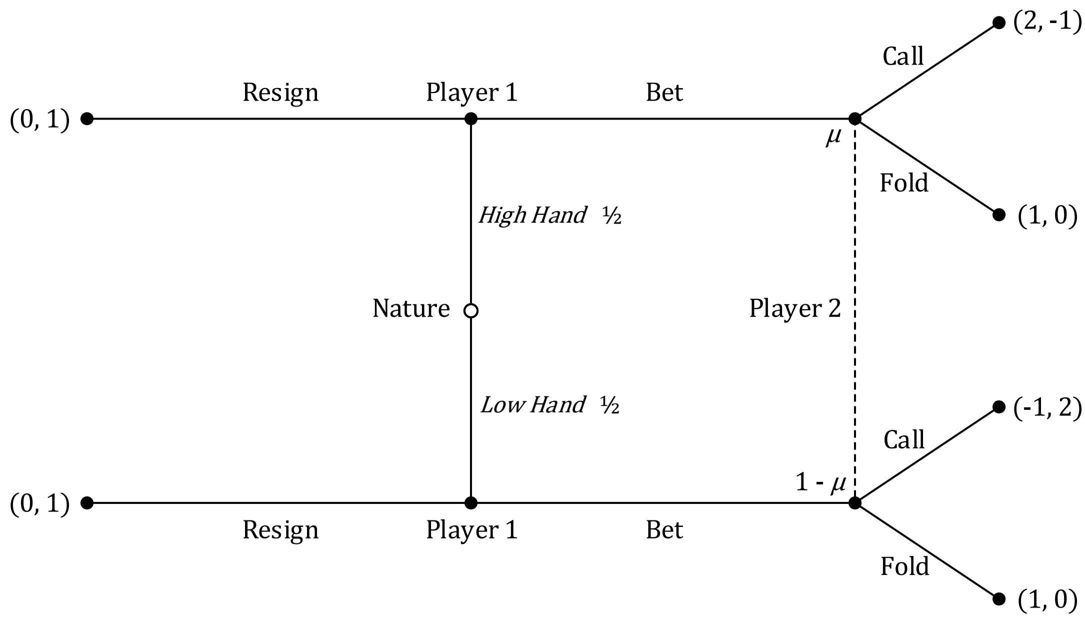
ECONS 424 – STRATEGY AND GAME THEORY
HANDOUT ON PERFECT BAYESIAN EQUILIBRIUM- III

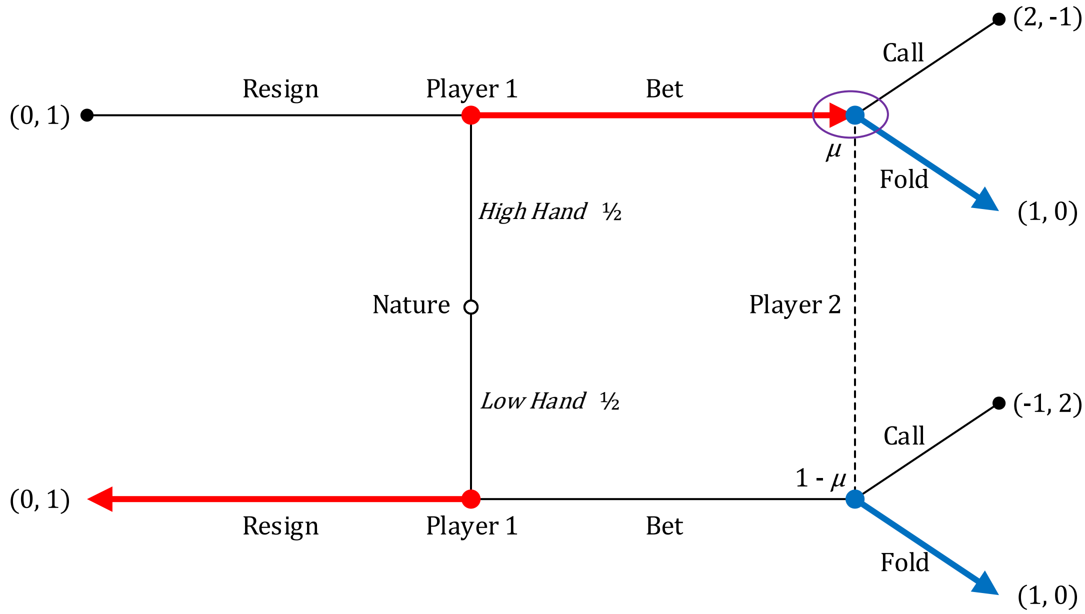
Semi-Separating equilibrium

Let us consider the following sequential game with incomplete information. Two players are playing the following super-simple poker game. The first player gets his cards, and obviously he is the only one who can observe them. He can either get a High hand or a Low hand, with equal probabilities. After observing his hand, he must decide whether to Bet (let us imagine that he is betting a fixed amount of a dollar), or Resign from the game. If he resigns, both when he has a High and a Low hand, his payoff is zero, and player 2 gets a dollar.

If he bets, player 2 must decide whether to Call or Fold from the game. Obviously, player makes his choice without being able to observe player 1's hands, but only observing that player 1 Bets.



1. ***Separating PBE where player 1 Bets when having a High hand, and Resigns when having a Low hand: (Bet, Resign) – Reasonable!***



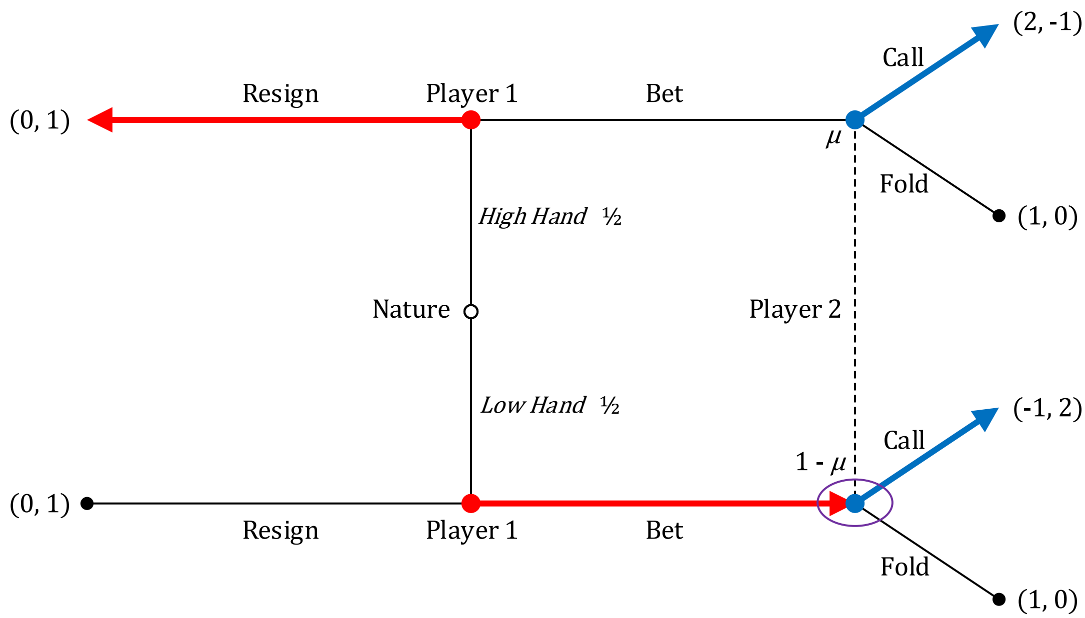
a) Player 2's beliefs (responder beliefs) in this separating PBE
- After observing Bet, μ=1, intuitively indicating that P2 believes that a Bet can only originate from a P1 with a High hand. (Graphically, this implies that P2 believes to be in the upper node of the information set.)

b) Given player 2's beliefs, which is player 2's optimal action, after observing that player 1 bets?
- Given the above beliefs, P2's optimal response is to Fold, since 0>-1.
  - You can shade the Fold branch for P2, for both types of P1 (something P2 cannot distinguish).

c) Given the previous points, what is player 1's optimal action (whether to Resign or Bet) when he has a High hand? What is his optimal action when having a Low hand?
- When P1 has a High hand, P1 prefers to Bet (as prescribed in this strategy profile) since his payoff from doing so (1) exceeds that from Resigning (0).
- When P1 has a Low hand, P1 prefers to Bet (deviating from the prescribed strategy profile) since his payoff from doing so (1) exceeds that from Resigning (0).

d) Can this separating strategy profile be supported as a PBE from your answer in *c)*?
- No, because P1 prefers to Bet regardless of his hand, contradicting the separating strategy profile (Bet, Resign).

2. *Separating PBE where player 1 Resigns when having a High hand, and Bets when having a Low hand: (Resign, Bet) – Crazy!*



a) Player 2's beliefs (responder beliefs) in this separating PBE

- After observing Bet, μ=0, intuitively indicating that P2 believes that a Bet can only originate from a P1 with a Low hand.
  - (Graphically, this implies that P2 believes to be in the lower node of the information set.)

b) Given player 2's beliefs, which is player 2's optimal action, after observing that player 1 bets?

- Given the above beliefs, P2's optimal response is to Call, since 2>0.
  - You can shade the Call branch for P2, for both types of P1 (something P2 cannot distinguish).

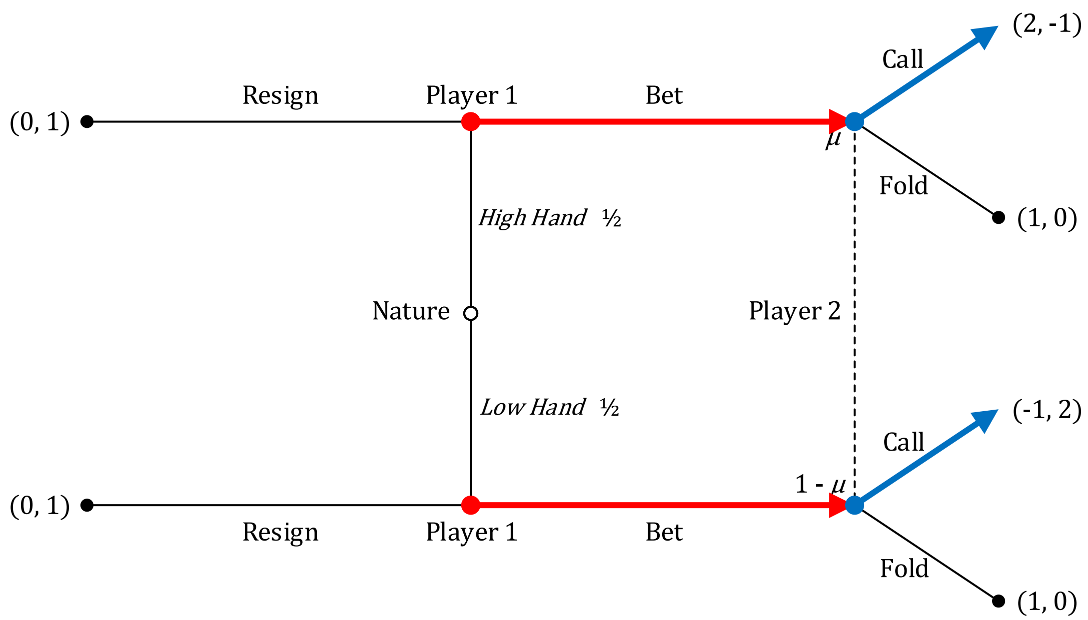
c) Given the previous points, what is player 1's optimal action (whether to Resign or Bet) when he has a High hand? What is his optimal action when having a Low hand?

- When P1 has a High hand, P1 prefers to Bet (deviating from the prescribed strategy profile) since his payoff from doing so (2, given that he anticipates P2 will Call) exceeds his payoff from Resigning (0).
- When P1 has a Low hand, P1 prefers to Resign (deviating from the prescribed strategy profile) since his payoff from doing so (0) exceeds his payoff from Betting (-1).

d) Can this separating strategy profile be supported as a PBE from your answer in *c)*?

- No, since P1 has incentives to deviate from the prescribed strategy profile, both when he has a High hand (he prefers to Bet) and when he has a Low hand (he prefers to Resign).

3. Pooling PBE with both types of player 1 playing Bet: (Bet, Bet)



a) Player 2's beliefs (responder beliefs) in this pooling PBE

- After observing Bet, his beliefs are

$$\mu = \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + \frac{1}{2} \times 1} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

- Intuitively indicating that P2's beliefs coincide with the prior probability distribution over types, i.e., $\mu = \frac{1}{2}$.

b) Given player 2's beliefs, which is player 2's optimal action, after observing that player 1 bets?

- In order to examine which is P2's response to Bet, we must separately find his expected utility from Call and Fold, as follows

$$EU_2(Call) = \frac{1}{2}(-1) + \frac{1}{2}(2) = \frac{1}{2}$$

$$EU_2(Fold) = \frac{1}{2}(0) + \frac{1}{2}(0) = 0$$

Implying that P2 prefers to Call since his expected utility is higher.

  - You can shade the Call branch for P2, for both types of P1 (something P2 cannot distinguish).

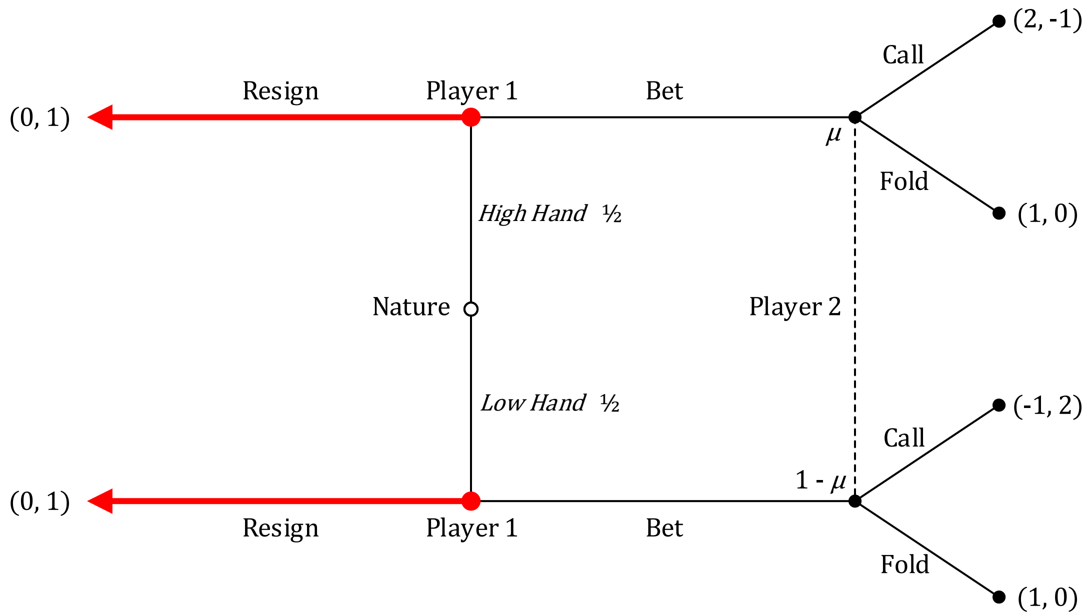
c) Given the previous points, what is player 1's optimal action (whether to Resign or Bet) when he has a High hand? What is his optimal action when having a Low hand?

- When P1 has a High hand, P1 prefers to Bet (as prescribed in this strategy profile) since his payoff from doing so (2, given that he anticipates P2 will Call) exceeds his payoff from Resigning (0).
- When P1 has a Low hand, P1 prefers to Resign (deviating from the prescribed strategy profile) since his payoff from doing so (0) exceeds his payoff from Betting (-1).

d) Can this pooling strategy profile with both types of player 1 playing Bet be supported as a PBE from your answer in *c)*?

- No, because P1 prefers to Resign when he has a Low hand, as described in the previous point, contradicting the prescribed pooling strategy profile where both types of P1 Bet.

4. Pooling PBE with both types of player 1 playing Resign: (Resign, Resign)



a) Player 2's beliefs (responder beliefs) in this pooling PBE

- After observing Bet (something that occurs off-the-equilibrium path, as indicated in the figure), P2's beliefs are

$$\mu = \frac{\frac{1}{2} \times 0}{\frac{1}{2} \times 0 + \frac{1}{2} \times 0} = \frac{0}{0}$$

and thus beliefs must be left undefined, i.e., $\mu \in [0, 1]$.

b) Given player 2's beliefs, which is player 2's optimal action, after observing that player 1 bets?

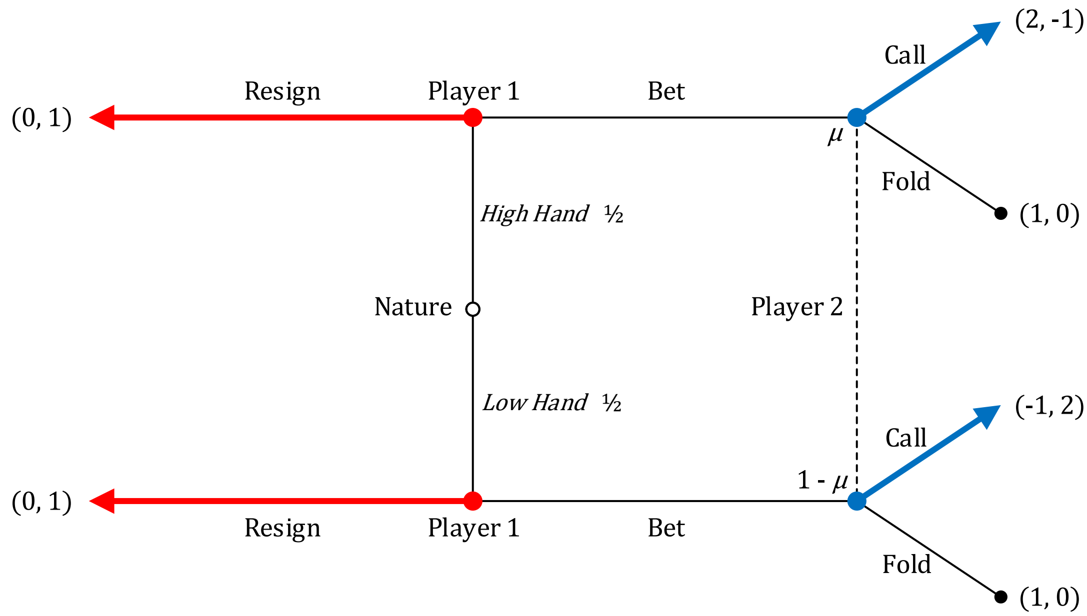
- In order to examine which is P2's response to Bet, we must separately find his expected utility from Call and Fold, as follows

$$EU_2(Call) = \mu(-1) + (1 - \mu)(2) = 2 - 3\mu$$
$$EU_2(Fold) = \mu(0) + (1 - \mu)(0) = 0$$

Implying that P2 prefers to Call if and only if $2 - 3\mu > 0$, or $\mu < \frac{2}{3}$.

- Hence, we will need to divide our subsequent analysis into two cases:
  - **Case 1:** $\mu < \frac{2}{3}$, entailing that P2 responds Calling if he observes Bet.
  - **Case 2:** $\mu > \frac{2}{3}$, entailing that P2 responds Folding if he observes Bet.

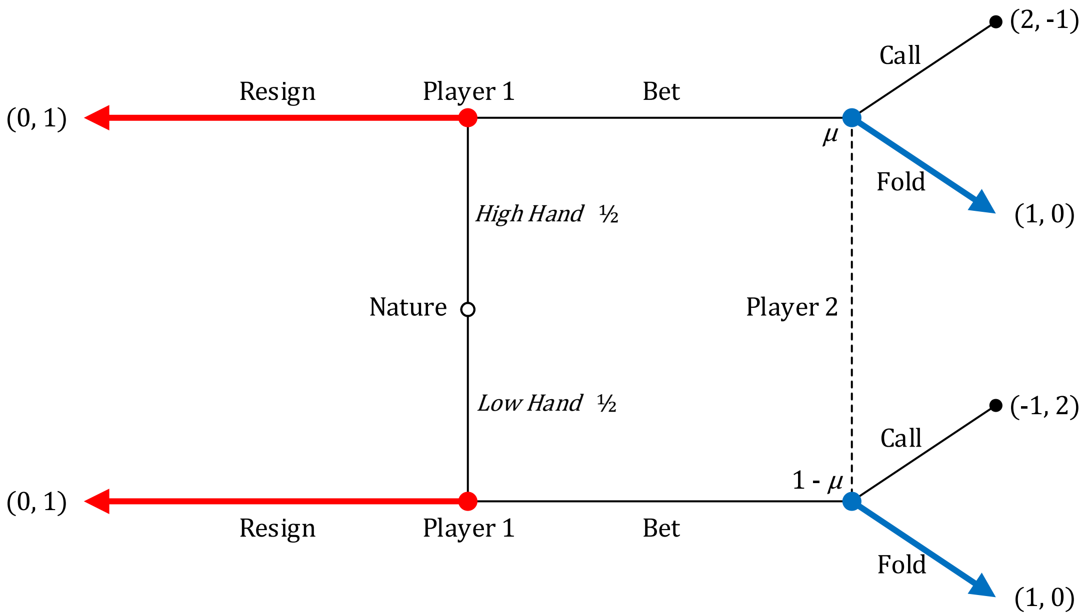
CASE 1: $\boldsymbol{\mu < \frac{2}{3}}$



Let us now check if this pooling strategy profile can be sustained as a PBE in this case ($\boldsymbol{\mu < \frac{2}{3}}$):

- When P1 has a High hand, he prefers to Bet, since his payoff from doing so (2) is larger than that from Rosining (0).
- We don't even need to check the case in which P1 has a Low hand, since the above argument already shows that the pooling strategy profile cannot be sustained as a PBE when $\mu < \frac{2}{3}$.

CASE 2: $\boldsymbol{\mu > \frac{2}{3}}$



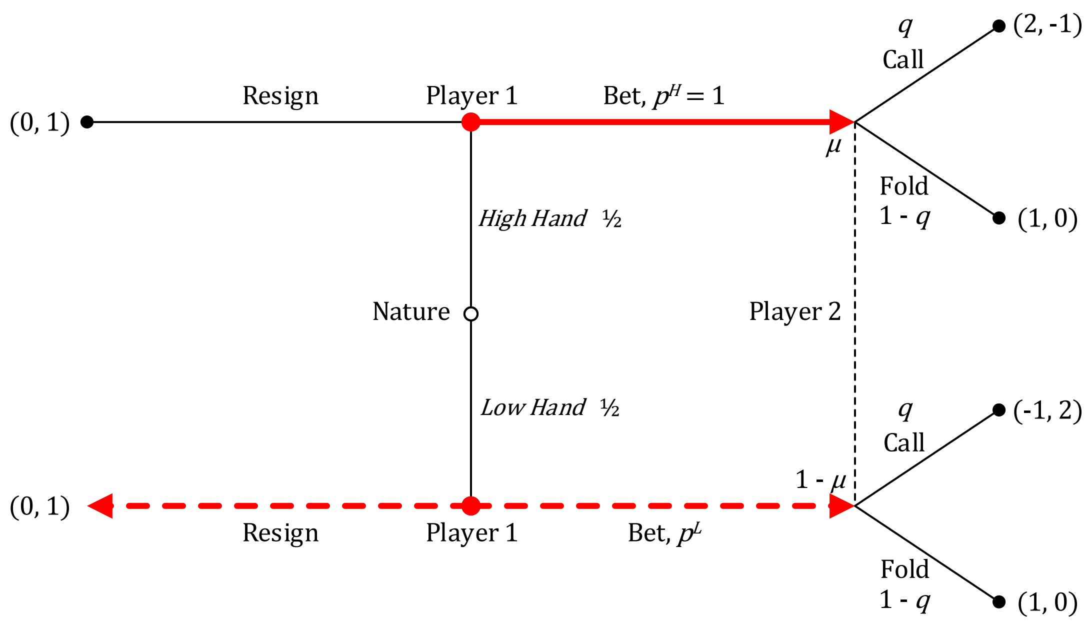
Let us now check if this pooling strategy profile can be sustained as a PBE in this case ($\boldsymbol{\mu > \frac{2}{3}}$):

- When P1 has a High hand, he prefers to Bet, since his payoff from doing so (1) is larger than that from Rosining (0).
- We don't even need to check the case in which P1 has a Low hand, since the above argument already shows that the pooling strategy profile cannot be sustained as a PBE when $\mu > \frac{2}{3}$.

Then, the pooling strategy profile where P1 Bets both when his hand is High and Low cannot be sustained as a PBE, regardless of P2's off-the-equilibrium beliefs.

- Notice that this implies that we have ruled out all pure strategy equilibria (either separating or pooling) and we must now move to check for semi-separating equilibria.

5. ***SEMI-Separating PBE where player 1 bets when having a High hand, and bets with a certain probability when having a Low hand: (Bet, Bet in mixed strategies)***
   - First, note that Bet is a strictly dominant strategy for P1 when he has a High hand. Indeed, his payoff from doing so (either 2 or 1, depending on whether P2 calls or folds) is strictly higher than his payoff from Resigning.
   - However, Bet is not necessarily a dominant strategy for P1 when he has a Low hand. He prefers to Bet if he anticipates that P2 will Fold, but prefers to Resign if P2 Calls.
   - Intuitively, P2 has incentives to call a Bet originating from a P1 with a Low hand. As a consequence, P1 doesn't want to convey his type to P2, but instead to conceal it so that P2 Folds. The way in which P1 can conceal his type is by randomizing his Betting strategy, as described in the figure.



a) What are player 2's beliefs (μ) that support the fact that player 2 is mixing? That is, what is the value of μ that makes player 2 indifferent between Call and Fold?
   - Player 2 must be mixing. If he If he wasn't, player 1 could anticipate his action and play pure strategies as in any of the above strategy profiles (which are not PBE of the game, as we just showed). Hence, player 2 must be indifferent between Call and Fold, as follows

$$\mu(-1) + (1-\mu)2 = 0$$

     which implies that $\mu = \frac{2}{3}$.
   - Hence, P2's beliefs in this semi-separating PBE must satisfy $\mu = \frac{2}{3}$.

b) Given player 2's beliefs, write Bayes' rule, taking into account that player 1 will always bet when having a High hand, but that he will mix (with probability p_L) when having a Low hand.

- Now, we must use player 2's beliefs that we found in the previous step, $\mu = \frac{2}{3}$, in order to find what mixed strategy player 1 uses. For that, we use Bayes' rule as follows:

$$\mu = \frac{2}{3} = \frac{\frac{1}{2} \times p^H}{\frac{1}{2} \times p^H + \frac{1}{2} \times p^L}$$

  where p^Hdenotes the probability with which P1 Bets when he has a High hand, and similarly p^L represents the probability that P1 Bets when he has a Low hand. Since we know that $p^H = 1$ given that P1 always Bets when he has a High hand (he Bets using pure strategies), then the above ratio becomes

$$\frac{2}{3} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2} \times p^L}$$

c) What is the value of p_L that satisfies the expression you wrote in question *b)*?

- Solving for the only unknown in this equality, p^L, we obtain $p^L = \frac{1}{2}$.
  - Hence, at this stage of our solution we know everything regarding player 1: he Bets with probability $p^L = \frac{1}{2}$ when he has a Low hand and Bets using pure strategies (with 100% probability) when he has a High hand, i.e., $p^H = 1$.

d) What is player 2's probability of calling (let us denoted by *q*) makes player 1 indifferent between Bet and Resign when he has a Low hand?

- If player 1 mixes with probability $p^L = \frac{1}{2}$ when he has a Low hand, it must be that player 2 makes him indifferent between Betting and Resigning. That is,

$$EU_1(Bet|Low) = EU_1(Resign|Low)$$
$$q(-1)+(1-q)1=0$$

  Solving for q, we obtain that P2 randomizes with probability $q = \frac{1}{2}$, i.e., he Calls with probability 50%. (Notice that now we are done: from questions *b* and *c* we had all the information we needed about P1's behavior, while from question *d* we obtained all necessary information about P2's actions.)

e) Summarize, with all your previous results, the Semi-Separating PBE of this poker game.

- Player 1 Bets when he has a High hand with full probability, $p^H = 1$, whereas he Bets when he has a Low hand with probability $p^L = \frac{1}{2}$.
- Player 2 Calls with probability $q = \frac{1}{2}$, and his beliefs are $\mu = \frac{2}{3}$.