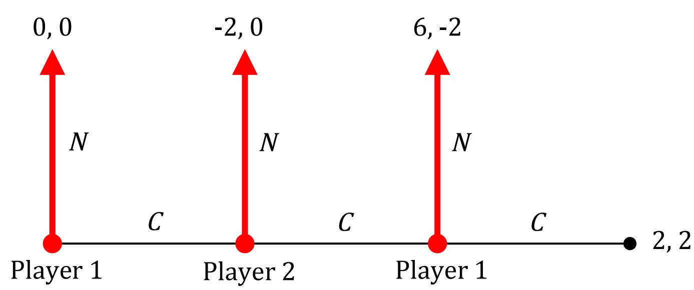
EconS 424- Strategy and Game Theory
Reputation and Incomplete information in a public good project*
How to find Semi-separating equilibria?

April 14, 2014

1 A public good game

Let us consider the following public good game, based on Watson (page 353), where two players sequentially contribute to a public good. First, player 1 decides to contribute to the public good (C) or not (N), afterwards player 2 responds to player 1's donation by contributing (C) or not (N), and finally player 1 is again called to move if player 2 contributes.

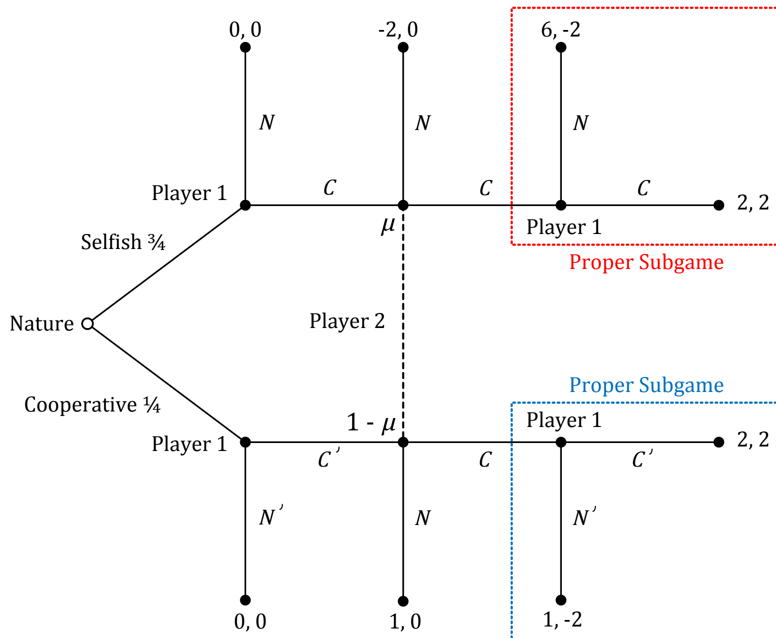


Sequential game with complete information.

Clearly, this a sequential game of complete information, which can be easily solved by using backward induction. Hence, the subgame perfect equilibrium of this game is (NN,N) where player 1 never contributes to the public good in the information sets in which he is called to move, and similarly player 2 does not contribute to the public good in the only node he is called to move. As a consequence, players' equilibrium payoffs are (0, 0). However, note that this result is inefficient, since players would benefit from the public good being provided, yielding (2, 2). Nonetheless, as we know from the notion of sequential rationality, every player expects all other players being rational along all the information sets of the game. This, in particular, makes player 2 expect that player 1 will not contribute to the public good in the first and last stages of the game, and similarly for player 1 regarding player 2's actions in the second stage of the game tree.

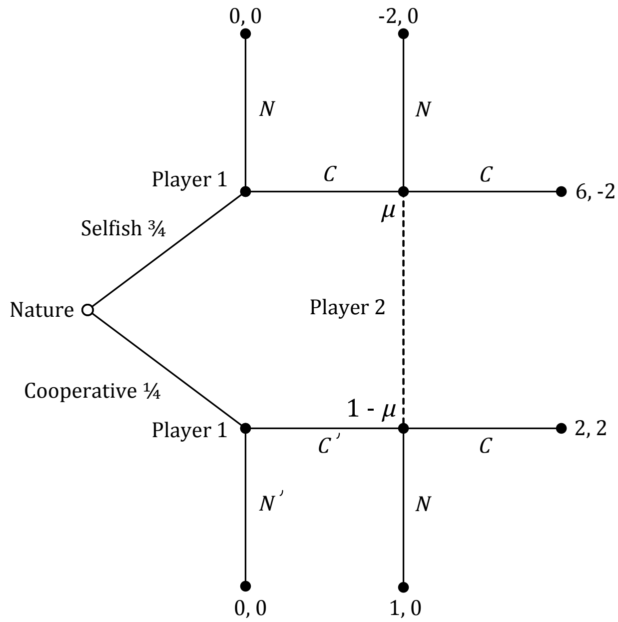
*Félix Muñoz-García, School of Economic Sciences, Washington State University, 103G Hulbert Hall, Pullman, WA. E-mail: fmunoz@wsu.edu.

As we next analyze, however, this unfortunate result can be avoided if players interact in an incomplete information environment (incomplete information game). In the figure below, we represent the same sequential-move game that was depicted above, but adding an element of incomplete information for player 2. Specifically, player 2 does not know whether player 1 is a "Selfish" type (who tries to free-ride player 2's donation and thus avoids giving to the public good), or a "Cooperative" type who always prefers to contribute to the public good, regardless of player 2's actions.

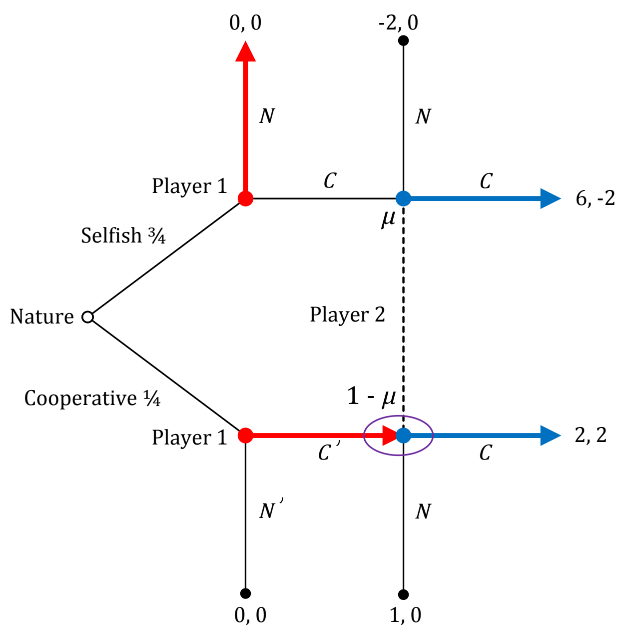


Introducing incomplete information

Let us now find the Perfect Bayesian Equilibria (PBE) of this sequential-move game of incomplete information by checking the existence of separating and pooling PBE, using the usual steps we described in class. In any case, since the last information set in which player 1 is called to move can be identified as a proper subgame of this game tree, we can apply backward induction at the third stage of the game, what simplifies the above sequential-move game to the following figure.



1.1 Separating PBE (N, C')



1. **Player 2's beliefs:** in this separating strategy profile P2's beliefs are $\mu = 0$. Intuitively, if P2 ever observes a contribution from P1, such a contribution must originate from the cooperative type. Graphically, this implies that P2 focuses on the lower node along the information set.
2. **Player 2:** Player 2 chooses C since $\mu = 0$ and $2 > 1$. Graphically, you can shade the C branch for P2, both after the lower node is reached and after the upper node is reached (since P2 cannot select a different strategy for each type of P1, given that he cannot distinguish P1's type).

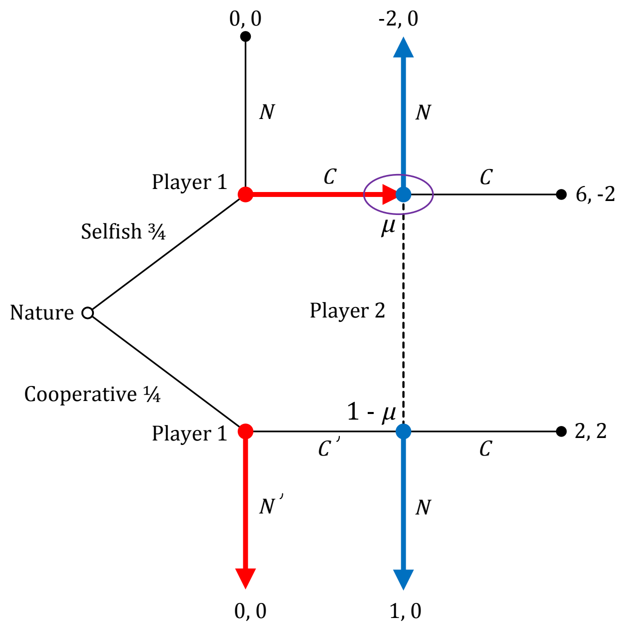
3. **Player 1:**

   (a) When being selfish, P1 chooses C since he anticipates that P2 contributes afterwards, yielding a payoff of 6 for P1, rather than choosing N, which only yields a payoff of 0. [This already shows that the suggested separating strategy profile cannot be sustained as a PBE of the game, since P1 has incentives to deviate from N to C when his type is selfish.]

   (b) When being cooperative, P1 chooses C' since he anticipates that P2 contributes afterwards, yielding a payoff of 2 for P1, rather than choosing N', which only yields a payoff of 0.

4. Hence, this separating strategy profile —where P1 contributes only when he is cooperative— *cannot* be supported as a PBE of this game, since both types of P1 contributes.

1.2 Separating PBE (C, N')



1. **Player 2's beliefs:** in this separating strategy profile P2's beliefs are $\mu = 1$. Intuitively, if P2 ever observes a contribution from P1, such a contribution must originate from the selfish type (I know, this is crazy). Graphically, this implies that P2 focuses on the upper node along the information set.

2. **Player 2:** Player 2 chooses N since $\mu = 1$ and $0 > -2$. Graphically, you can shade the N branch for P2, both after the upper node is reached and after the lower node is reached (since P2 cannot select a different strategy for each type of P1, given that he cannot distinguish P1's type).

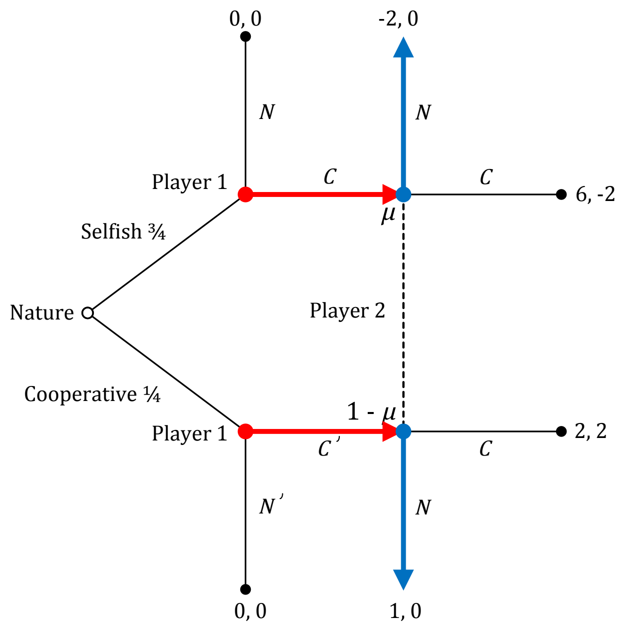
3. **Player 1:**

   (a) When being selfish, P1 chooses N, yielding a payoff of 0, rather than cooperating, which yields a payoff of -2 (given that he anticipates that P2 does not contribute afterwards). [This already shows that the suggested separating strategy profile cannot be sustained as a PBE of the game, since P1 has incentives to deviate from C to N when his type is selfish.]

   (b) When being cooperative, P1 chooses C' since his payoff from doing so, 1 given that he anticipates that P2 contributes afterwards, exceeds that of choosing N', which only yields a payoff of 0.

4. Hence, this separating strategy profile —where P1 contributes only when he is selfish— *cannot* be supported as a PBE of this game, since P1 does not have incentives to contribute when his type is selfish, as shown in the point 3(a) above.

1.3 Pooling PBE (C, C')



1. **Player 2's beliefs:**

$$\mu = \frac{\frac{3}{4}p_{self}}{\frac{3}{4}p_{self} + \frac{1}{4}p_{coop}} = \frac{\frac{3}{4} \times 1}{\frac{3}{4} \times 1 + \frac{1}{4} \times 1} = \frac{3}{4}$$

   where p_{self} denotes the probability with which the selfish type contributes, whereas p_{coop} represents the probability that the cooperative type contributes. In this pooling strategy profile where both types contribute with 100%, these probabilities satisfy $p_{self} = p_{coop} = 1$, which implies that P2's beliefs, μ, coincide with the prior probability distribution, $\frac{3}{4}$.

   - Intuitively, P2 cannot infer any additional information from P1's type after observing that he contributes, since both types of P1 contribute in this pooling strategy profile.

2. **Player 2:** Player 2 expected utility levels from contributing and not contributing are, respectively

$$\begin{aligned} EU_2(C) &= \frac{3}{4}(-2) + \frac{1}{4}(2) = -1 \\ EU_2(N) &= \frac{1}{4}0 + \frac{3}{4}0 = 0 \end{aligned}$$

   and hence player 2 chooses not to contribute (N). Graphically, you can shade the N branch for P2, both after the upper node is reached and after the lower node is reached (since P2 cannot select a different strategy for each type of P1, given that he cannot distinguish P1's type).

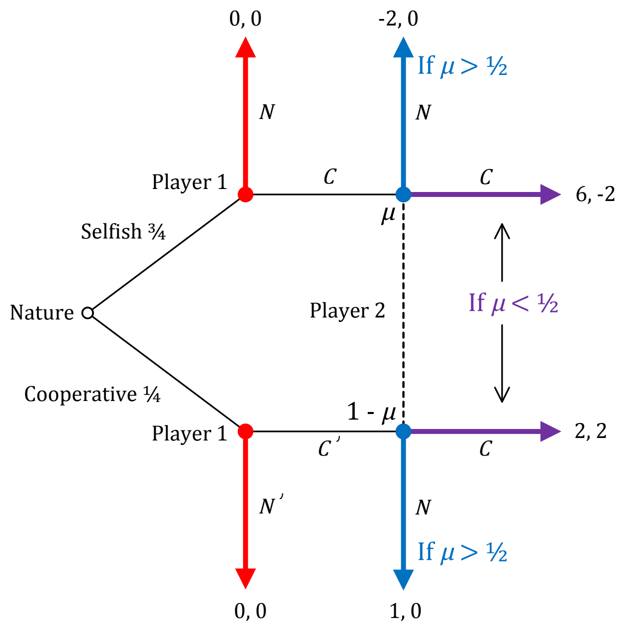
3. **Player 1:**

   (a) When being selfish, P1 chooses N, yielding a payoff of 0, rather than cooperating, which yields a payoff of -2 (given that he anticipates that P2 does not contribute afterwards). [This already shows that the suggested pooling strategy profile cannot be sustained as a PBE of the game, since P1 has incentives to deviate from C to N when his type is selfish.]

(b) When being cooperative, P1 chooses C', since his payoff from doing so (1) given that he anticipates that P2 contributes afterwards, exceeds that of choosing N', which only yields a payoff of 0.

4. Hence, this pooling strategy profile —where both types of P1 contribute— *cannot* be supported as a PBE of this game, since P1 does not have incentives to contribute when his type is selfish, as shown in the point 3(a) above.

1.4 Pooling PBE (N, N')



1. **Player 2's beliefs:** Note that player 2's information set is not reached in equilibrium, since both types of P1 choose not to contribute, as represented in the figure. Hence, player 2's beliefs, μ, are

$$\mu = \frac{\frac{3}{4}p_{self}}{\frac{3}{4}p_{self} + \frac{1}{4}p_{coop}} = \frac{\frac{3}{4}0}{\frac{3}{4}0 + \frac{1}{4}0} = \frac{0}{0}$$

where $p_{self} = p_{coop} = 0$ since no type of P1 cooperates. P2's beliefs must then be left undefined, i.e., $\mu \in [0, 1]$.

2. **Player 2:** Player 2 expected utility levels from contributing and not contributing are, respectively

$$\begin{aligned} EU_2(C) &= \mu(-2) + (1-\mu)(2) = 2 - 4\mu \\ EU_2(N) &= \mu 0 + (1-\mu)0 = 0 \end{aligned}$$

and hence player 2 chooses to contribute if and only if $2 - 4\mu > 0$. That is, he contributes if $\mu < \frac{1}{2}$. This implies that we will have to divide our following analysis into two cases:

- Case 1: $\mu < \frac{1}{2}$, implying that P2 responds *contributing* if he observes an (off-the-equilibrium) contribution from P1.
- Case 2: $\mu > \frac{1}{2}$, implying that P2 responds *not contributing* if he observes an (off-the-equilibrium) contribution from P1.

3. **Player 1:**

   (a) CASE 1: $\mu < \frac{1}{2}$.

      i. When being selfish, P1 chooses C since he anticipates that P2 contributes afterwards, yielding a payoff of 6, rather than choosing N, which only yields a payoff of 0. [This already shows that the suggested pooling strategy profile cannot be sustained as a PBE of the game when $\mu < \frac{1}{2}$, since P1 has incentives to deviate from N to C when his type is selfish.]

   ii. When being cooperative, P1 chooses C' since he anticipates that P2 contributes afterwards, yielding a payoff of 2 for P1, rather than choosing N', which only yields a payoff of 0.
   iii. Hence, this pooling strategy profile —where no type of P1 contributes— *cannot* be supported as a PBE of this game when $\mu < \frac{1}{2}$, since both types of P1 has incentives to contribute.

(b) CASE 2: $\mu > \frac{1}{2}$.

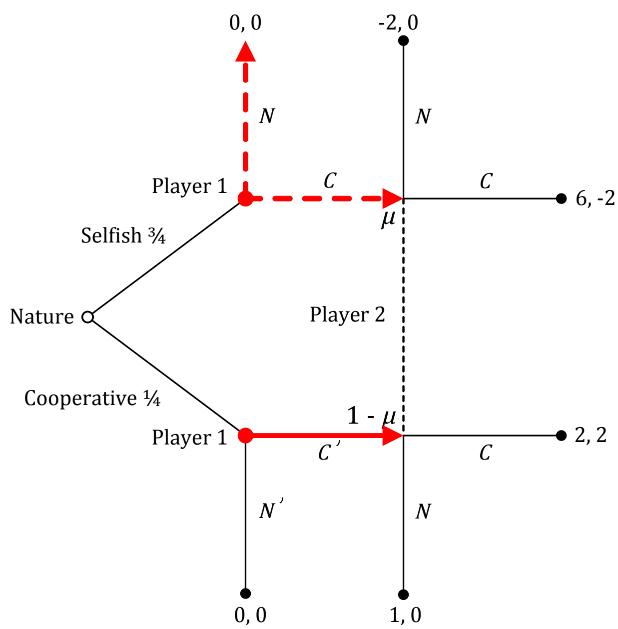
   i. When being selfish, P1 chooses N, yielding a payoff of 0, rather than cooperating, which yields a payoff of -2 (given that he anticipates that P2 does not contribute afterwards).
   ii. When being cooperative, P1 chooses C', since his payoff from doing so (1) given that he anticipates that P2 contributes afterwards, exceeds that of choosing N', which only yields a payoff of 0.
   iii. Hence, this pooling strategy profile —where no type of P1 contributes— *cannot* be supported as a PBE of this game when $\mu > \frac{1}{2}$ either, since P1 has incentives to contribute when being cooperative.

4. Summarizing, this pooling strategy profile —where no type of P1 contributes— *cannot* be supported as a PBE of this game since either or both types of P1 has incentives to deviate towards contributions to the public good.

1.5 Semi-Separating PBE

We have just showed that P1 cannot be using pure strategies. He must be using mixed strategies. The figure below depicts a strategy profile where P1 mixes between contributing and not contributing to the public good when his type is selfish (dashed lines), but contributes using pure strategies (100% of the times) when his type is cooperative. Intuitively, for the cooperative contributing (C') strictly dominates not contributing (N') regardless of P2's response. In particular, the payoff he obtains after C', either 2 or 1, is larger than his payoff from selecting N', 0. In contrast, the selfish type of P1 prefers to contribute (C) only if P2 contributes afterwards (yielding a payoff of 6).

If P1 anticipates that P2 won't contribute, his best response is to select N in the first stage of the game. Essentially, the selfish type wants to induce P2's contribution but "concealing" his type. Indeed, if P2 could perfectly infer that P1's contribution comes from a selfish type, P2 would not contribute (since 0>-2).



1. **Player 2's beliefs:** Player 2 must be mixing. If he wasn't, player 1 could anticipate his response and play pure strategies as in any of the above strategy profiles (which are not PBE of the game, as we just showed). Hence, if player 2 mixes he must be indifferent between contributing and not contributing to the public good:
$$EU_2(C) = EU_2(N)$$
$$\mu(-2) + (1-\mu)(2) = \mu 0 + (1-\mu)0 \implies \mu = \frac{1}{2}$$
Hence, player 2's beliefs in this semi-separating PBE must satisfy $\mu = \frac{1}{2}$.

2. **Using Bayes' rule to determine P1's probabilities:** Now, we must use the beliefs of player 2 that we found in the previous step, $\mu = \frac{1}{2}$, in order to find what is the mixed strategy that player 1 uses. For that, we use Bayes' rule as follows:
$$\mu = \frac{1}{2} = \frac{\frac{3}{4}p_{Self}}{\frac{3}{4}p_{Self} + \frac{1}{4}p_{Coop}}$$

But we know that $p_{Coop} = 1$ since player 1 always contributes when he is a Cooperative type. Hence, the above ratio becomes

$$\frac{1}{2} = \frac{\frac{3}{4}p_{Self}}{\frac{3}{4}p_{Self} + \frac{1}{4}}$$

and solving for the only unknown in this equality, p_{Self}, we obtain $p_{Self} = \frac{1}{3}$, which is the probability with which the Selfish type of player 1 contributes to the public good.

- Hence, at this stage of our solution we know everything regarding player 1: He contributes to the public good with probability $p_{Self} = \frac{1}{3}$ when he is the Selfish type, whereas he contributes using pure strategies (with 100% probability) when he is the Cooperative type, i.e., $p_{Coop} = 1$.

3. **Player 2's probabilities:** If player 1 mixes with probability $p_{Self} = \frac{1}{3}$ when he is a Selfish type, it must be that player 2 makes him indifferent between contributing and not contributing to the public good. (Recall that this is one of the interpretations for a player to use mixed strategies: to make the other player unable to anticipate his moves). More formally, if a selfish P1 is indifferent between C and N,

$$EU_1(C|Self) = EU_1(N|Self)$$

$$r6 + (1-r)(-2) = 0$$

where r denotes the probability with which player 2 mixes between contributing and not contributing. Solving for r, we obtain $r = \frac{1}{4}$. (Notice that now we are done: from point 2 above we had all the information we needed about P1's behavior, while from point 3 we obtained all necessary information about P2's actions. In the next point we just need to summarize our results).

4. Hence, this strategy profile can be supported as a Semi-Separating PBE of this game where:

   (a) Player 1 contributes to the public good with probability $p_{Self} = \frac{1}{3}$ when he is a Selfish type, whereas he contributes with full probability $p_{Coop} = 1$ when he is a Cooperative type.

   (b) Player 2 contributes to the public good with probability $r = \frac{1}{4}$; and his beliefs are $\mu = \frac{1}{2}$.

Summarizing, even if the probability of dealing with a selfish type is relatively low (here $\frac{1}{4}$, but it could be lower), the public project has a positive probability of being built. In particular, the selfish type of P1 contributes to it with probability $p_{Self} = \frac{1}{3}$ and the uninformed P2 responds contributing with probability $r = \frac{1}{4}$.