From Bayesian Nash Equilibrium (BNE) to Perfect Bayesian Equilibrium (PBE)

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So far we have learned how to find BNEs in incomplete information games. We are doing great!

In settings where players are uncertain about their opponent’s types... this is a fantastic solution concept.

since it specifies optimal strategies, given the information every player has access to.
BNEs and Sequential rationality

- What if players interact in a sequential-move game?
  - Can the BNE prescribe "insensible" behavior? Yes!
  - But, what do we mean by "insensible" behavior?
    - Strategies that are not sequentially rational.
    - We will, hence, need a solution concept that guarantees sequential rationality (as SPNE, but applied to contexts of incomplete information).

- Let's show this with an example.

- Use now the separate handout:
  - "Why do we need Perfect Bayesian Equilibrium? Asking for sequential rationality in sequential-move games with incomplete information."
More examples about how to find PBEs

- After finding the PBEs in the Gift game...
- Let’s now practice with another example (*Monetary Authority game*):
  - Now we will consider a Strong or Weak monetary authority, who makes an inflation announcement.
  - And a labor union (uninformed about the monetary authority’s true commitment with low inflation policies, either Strong or Weak)... 
  - decides whether to demand large or moderate salary increases.
Example: Let us consider the following sequential game with incomplete information:

- A monetary authority (such as the Federal Reserve Bank) privately observes its real degree of commitment with maintaining low inflation levels.
- After knowing its type (either Strong or Weak), the monetary authority decides whether to announce that the expectation for inflation is either High or Low.
- A labor union, observing the message sent by the monetary authority, decides whether to ask for high or low salary raises (denoted as H or L, respectively)
The game tree that represents this incomplete information game is, hence, as follows:
Before starting to find all possible PBEs...

- Let us briefly set up our "road map"

That is, let’s recall the 5-step procedure that we need to follow in order to find PBEs.
Procedure to find PBEs

1. Specify a strategy profile for the privately informed player, either separating or pooling.
   - In our above example, there are only four possible strategy profiles for the privately informed monetary authority: two separating strategy profiles, $High^S Low^W$ and $Low^S High^W$, and two pooling strategy profiles, $High^S High^W$ and $Low^S Low^W$.
   - (For future reference, it might be helpful to shade the branches corresponding to the strategy profile we test.)

2. Update the uninformed player’s beliefs using Bayes’ rule, whenever possible.
   - In our above example, we need to specify beliefs $\mu$ and $\gamma$, which arise after the labor union observes a high or a low inflation announcement, respectively.
3. Given the uninformed player’s updated beliefs, find his optimal response.

- In our above example, we first determine the optimal response of the labor union (H or L) upon observing a high-inflation announcement (given its updated belief $\mu$),
- we then determine its optimal response (H or L) after observing a low-inflation announcement (given its updated belief $\gamma$).

(Also for future reference, it might be helpful to shade the branches corresponding to the optimal responses we just found.)
4. Given the optimal response of the uninformed player, find the optimal action (message) for the informed player.
   - In our previous example, we first check if the Strong monetary authority prefers to make a high or low inflation announcement (given the labor union’s responses determined in step 3).
   - We then operate similarly for the Weak type of monetary authority.
5. Then check if this strategy profile for the informed player coincides with the profile suggested in step 1.

- If it coincides, then this strategy profile, updated beliefs and optimal responses can be supported as a PBE of the incomplete information game.
- Otherwise, we say that this strategy profile cannot be sustained as a PBE of the game.
Let us next separately apply this procedure to test each of the four candidate strategy profiles:

- two separating strategy profiles:
  - $High^S Low^W$, and
  - $Low^S High^W$.

- And two pooling strategy profiles:
  - $High^S High^W$, and
  - $Low^S Low^W$.  

Let us first check separating strategy profile: $Low^S High^W$.

**Step #1:** Specifying strategy profile $Low^S High^W$ that we will test.

(See shaded branches in the figure.)
Step #2: Updating beliefs

(a) After high inflation announcement (left-hand side)

\[ \mu = \frac{0.6 \alpha^{\text{Strong}}}{0.6 \alpha^{\text{Strong}} + 0.4 \alpha^{\text{Weak}}} = \frac{0.6 \times 0}{0.6 \times 0 + 0.4 \times 1} = 0 \]
Step #2: Updating beliefs

- This implies that after high inflation...
- the labor union restricts its belief to the lower left-hand corner (see box), since $\mu = 0$ and $1 - \mu = 1$
Step #2: Updating beliefs

(b) After low inflation announcement (right-hand side)

\[
\gamma = \frac{0.6 \left(1 - \alpha^{\text{Strong}}\right)}{0.6 \left(1 - \alpha^{\text{Strong}}\right) + 0.4 \left(1 - \alpha^{\text{Weak}}\right)} = \frac{0.6 \times 1}{0.6 \times 1 + 0.4 \times 0} = 1
\]
Separating equilibrium with (Low,High)

**Step #2: Updating beliefs**

- This implies that, after low inflation...
- the labor union restricts its belief to the upper right-hand corner (see box), since $\gamma = 1$ and $1 - \gamma = 0$. 
Step #3: Optimal response

(a) After high inflation announcement, respond with H since

\[ 0 > -100 \]

in the lower left-hand corner of the figure (see blue box).
Step #3: Optimal response

(b) After low inflation announcement, respond with L since

\[ 0 > -100 \]

in the upper right-hand corner of the figure (see box).
We can hence summarize the optimal responses we just found, by shading them in the figure:

- H after high inflation, but L after low inflation.

![Diagram showing separating equilibrium with (Low,High)](image-url)
Step #4: Optimal messages by the informed player

(a) When the monetary authority is Strong, if it chooses Low (as prescribed), its payoff is $300,
while if it deviates, its payoff decreases to $0.
(No incentives to deviate).
Step #4: Optimal messages

(b) When the monetary authority is Weak, if it chooses High (as prescribed), its payoff is $100, while if it deviates, its payoff decreases to $50. (No incentives to deviate either).
Since no type of privately informed player (monetary authority) has incentives to deviate,

- The separating strategy profile $Low^S High^W$ can be sustained as a PBE.
Separating equilibrium with (High,Low)

- Let us now check the opposite separating strategy profile: $High^S Low^W$.

Step #1: Specifying strategy profile $High^S Low^W$ that we will test.
  
  (See shaded branches in the figure.)
Step #2: Updating beliefs

(a) After high inflation announcement

\[ \mu = \frac{0.6 \alpha^{Strong}}{0.6 \alpha^{Strong} + 0.4 \alpha^{Weak}} = \frac{0.6 \times 1}{0.6 \times 1 + 0.4 \times 0} = 1 \]
Step #2: Updating beliefs

- Hence, after high inflation...
- the labor union restricts its beliefs to $\mu = 1$ in the upper left-hand corner (see box).
Separating equilibrium with (High, Low)

**Step #2:** Updating beliefs

(b) After low inflation announcement

\[
\gamma = \frac{0.6 \left(1 - \alpha^{Strong}\right)}{0.6 \left(1 - \alpha^{Strong}\right) + 0.4 \left(1 - \alpha^{Weak}\right)} = \frac{0.6 \times 0}{0.6 \times 0 + 0.4 \times 1} = 0
\]
Step #2: Updating beliefs

Hence, after low inflation...
the labor union restricts its beliefs to \( \gamma = 0 \) (i.e., \( 1 - \gamma = 1 \))
in the lower right-hand corner (see box).
Separating equilibrium with (High,Low)

**Step #3:** Optimal response

(a) After high inflation announcement, respond with L since

\[ 0 > -100 \]

in the upper left-hand corner of the figure (see box).
**Step #3:** Optimal response

(a) After low inflation announcement, respond with H since

\[
0 > -100
\]

in the lower right-hand corner of the figure (see box).
Summarizing the optimal responses we just found:
- L after high inflation, but H after high inflation.
Step #4: Optimal messages of the informed player

(a) When the monetary authority is Strong, if it chooses High (as prescribed), its payoff is $200, while if it deviates, its payoff decreases to $100. (No incentives to deviate).
Step #4: Optimal messages

(b) When the monetary authority is Weak, if it chooses Low (as prescribed), its payoff is $0,
while if it deviates, its payoff increases to $150.
(Incentives to deviate!!).
Since we found one type of privately informed player (the Weak monetary authority) who has incentives to deviate...

- The separating strategy profile $High^S Low^W$ cannot be sustained as a PBE.
Let us now test the pooling strategy profile $High^S High^W$.

- **Step #1:** Specifying strategy profile $High^S High^W$ that we will test.
  - (See shaded branches in the figure.)
Step #2: Updating beliefs

(a) After high inflation announcement

\[ \mu = \frac{0.6 \alpha^{Strong}}{0.6 \alpha^{Strong} + 0.4 \alpha^{Weak}} = \frac{0.6 \times 1}{0.6 \times 1 + 0.4 \times 1} = 0.6 \]

so the high inflation announcement is uninformative.
Step #2: Updating beliefs

(b) After low inflation announcement (off-the-equilibrium path)

\[
\gamma = \frac{0.6 \left(1 - \alpha^{Strong}\right)}{0.6 \left(1 - \alpha^{Strong}\right) + 0.4 \left(1 - \alpha^{Weak}\right)} = \frac{0.6 \times 0}{0.6 \times 0 + 0.4 \times 0} = 0
\]

hence, \( \gamma \in [0, 1] \).
Step #3: Optimal response

(a) After high inflation announcement (along the equil. path), respond with L since

\[ EU_{Labor} (H|High) = 0.6 \times (-100) + 0.4 \times 0 = -60 \]

\[ EU_{Labor} (L|High) = 0.6 \times 0 + 0.4 \times (-100) = -40 \]
Step #3: Optimal response

(a) After low inflation announcement (off-the-equil.),

\[ EU_{Labor}(H|\text{Low}) = \gamma \times (-100) + (1 - \gamma) \times 0 = -100\gamma \]
\[ EU_{Labor}(L|\text{Low}) = \gamma \times 0 + (1 - \gamma) \times (-100) = -100 + 100\gamma \]

i.e., respond with H if \( \gamma < \frac{1}{2} \).
Pooling equilibrium with (High,High)

- Summarizing the optimal responses we found...
  - Note that we need to divide our analysis into two cases:
  - **Case 1**, where $\gamma < \frac{1}{2}$, implying that the labor union responds with H after observing low inflation (right-hand side).
Pooling equilibrium with (High,High)

- and...

- **Case 2**, where $\gamma \geq \frac{1}{2}$, implying that the labor union responds with L after observing low inflation (right-hand side).
Pooling equilibrium with (High, High)

**Case 1**, where $\gamma < \frac{1}{2}$

- **Step #4**: Optimal messages
  - (a) When the monetary authority is Strong, if it chooses High (as prescribed), its payoff is $200$
  - while if it deviates, its payoff decreases to $100$
  - (No incentives to deviate)
Pooling equilibrium with (High,High)

**Case 1**, where $\gamma < \frac{1}{2}$

- **Step #4**: Optimal messages
  - (b) When the monetary authority is Weak, if it chooses High (as prescribed), its payoff is $150$,
  - while if it deviates, its payoff drops to $0$.
  - (No incentives to deviate either).
Poolung equilibrium with (High,High)

Case 1, where $\gamma < \frac{1}{2}$

- No type of monetary authority has incentives to deviate.
- Hence, the pooling strategy profile $High^S High^W$ can be sustained as a PBE when off-the-equilibrium beliefs satisfy $\gamma < \frac{1}{2}$. 
Pooling equilibrium with (High, High)

Case 2, where $\gamma \geq \frac{1}{2}$

- **Step #4**: Optimal messages
  - (a) When the monetary authority is Strong, if it chooses High (as prescribed), its payoff is $200,$
  - while if it deviates, its payoff **increases** to $300.$
  - (Incentives to deviate!!).
Case 2, where $\gamma \geq \frac{1}{2}$

- **Step #4**: Optimal messages
  - (b) When the monetary authority is Weak, if it chooses High (as prescribed), its payoff is $150$,
  - while if it deviates, its payoff drops to $50$.
  - (No incentives to deviate).
Pooling equilibrium with (High,High)

Case 2, where $\gamma \geq \frac{1}{2}$

- Since we found one type of privately informed player (the Strong monetary authority) who has incentives to deviate...
  - The pooling strategy profile $High^S High^W$ cannot be sustained as a PBE when off-the-equilibrium beliefs satisfy $\gamma \geq \frac{1}{2}$.
Let us now examine the opposite pooling strategy profile.

**Step #1:** Specifying strategy profile $Low^S Low^W$ that we will test.

(See shaded branches in the figure.)
Step #2: Updating beliefs

(a) After a low inflation announcement

\[
\gamma = \frac{0.6 \left( 1 - \alpha^{Strong} \right)}{0.6 \left( 1 - \alpha^{Strong} \right) + 0.4 \left( 1 - \alpha^{Weak} \right)} = \frac{0.6 \times 1}{0.6 \times 1 + 0.4 \times 1} = 0.6
\]

so posterior and prior beliefs coincide.
Step #2: Updating beliefs

(b) After a high inflation announcement (off-the-equil. path)

\[ \mu = \frac{0.6 \alpha_{Strong}}{0.6 \alpha_{Strong} + 0.4 \alpha_{Weak}} = \frac{0.6 \times 0}{0.6 \times 0 + 0.4 \times 0} = 0 \]

hence, \( \mu \in [0, 1] \).
Step #3: Optimal response

(a) After a low inflation announcement (along the equilibrium path), respond with L since

\[
EU_{Labor} (H|Low) = 0.6 \times (-100) + 0.4 \times 0 = -60
\]
\[
EU_{Labor} (L|Low) = 0.6 \times 0 + 0.4 \times (-100) = -40
\]
Step #3: Optimal response

(a) After a high inflation announcement (off-the-equil.),

\[
E_{Labor} (H|Low) = \mu \times (-100) + (1 - \mu) \times 0 = -100\mu
\]

\[
E_{Labor} (L|Low) = \mu \times 0 + (1 - \mu) \times (-100) = -100 + 100\mu
\]

i.e., respond with H if \(\mu < \frac{1}{2}\).
Pooling equilibrium with (Low, Low)

- Summarizing the optimal responses we found...
- Note that we need to divide our analysis into two cases:
- **Case 1**, where \( \mu < \frac{1}{2} \), implying that the labor union responds with H after observing high inflation (left-hand side).
Pooling equilibrium with \((\text{Low,Low})\)

- and...

- **Case 2**, where \(\mu \geq \frac{1}{2}\), implying that the labor union responds with \(L\) after observing high inflation (left-hand side).
Pooling equilibrium with (Low,Low)

**Case 1**, where $\mu < \frac{1}{2}$

- **Step #4**: Optimal messages
  - (a) When the monetary authority is Strong, if it chooses Low (as prescribed), its payoff is $300$,
  - while if it deviates, its payoff decreases to $200$.
  - (No incentives to deviate).
Pooling equilibrium with (Low,Low)

Case 1, where $\mu < \frac{1}{2}$

- **Step #4:** Optimal messages
  - (b) When the monetary authority is Weak, if it chooses High (as prescribed), its payoff is $50$, while if it deviates, its payoff **increases** to $100$. (Incentives to deviate!!)
Pool equilibrium with (Low,Low)

**Case 1**, where $\mu < \frac{1}{2}$

- Since we found one type of privately informed player (the Weak monetary authority) who has incentives to deviate...
  - The pooling strategy profile $Low^S Low^W$ **cannot** be sustained as a PBE when off-the-equilibrium beliefs satisfy $\mu < \frac{1}{2}$. 
Pooling equilibrium with (Low,Low)

Case 2, where $\mu \geq \frac{1}{2}$

- **Step #4:** Optimal messages
  - (a) When the monetary authority is Strong, if it chooses Low (as prescribed), its payoff is $300$,
  - while if it deviates, its payoff decreases to $200$.
  - (No incentives to deviate).
Case 2, where $\mu \geq \frac{1}{2}$

- **Step #4:** Optimal messages
  - (b) When the monetary authority is Weak, if it chooses Low (as prescribed), its payoff is $50$,
  - while if it deviates, its payoff increases to $150$.
  - (Incentives to deviate!!).
Pooling equilibrium with (Low,Low)

Case 2, where $\mu \geq \frac{1}{2}$

- Since we found one type of privately informed player (the Weak monetary authority) who has incentives to deviate...
  - The pooling strategy profile $Low^S Low^W$ cannot be sustained as a PBE when off-the-equilibrium beliefs satisfy $\mu \geq \frac{1}{2}$.
- Hence, the pooling strategy profile $Low^S Low^W$ cannot be sustained as a PBE for any off-the-equilibrium beliefs $\mu$. 