Introduction to Games and their Representation

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EconS 503

- Game Theory as the study of interdependence
  - "No man is an island"
- Definition:
  - Game Theory: a formal way to analyze **interaction** among a **group** of **rational** agents who behave strategically.

- Several important elemants of this definition help us understand what is game theory, and what is not:
- **Interaction:** If your actions do not affect anybody else, that is not a situation of interdependence.
- **Group:** we are not interested in games you play with your imaginary friend, but with other people, firms, etc.
- **Rational agents:** we assume that agents will behave rationally especially if the stakes are high and you allow them sufficient time to think about their available strategies.
  - Although we mention some experiments in which individuals do not behave in a completely rational manner...
  - these "anomalies" tend to vanish as long as you allow for sufficient repetitions, i.e., everybody ends up learning, or you raise stakes sufficiently (high incentives).

Examples:

- Output decision of two competing firms:
  - Cournot model of output competition.
- Research and Development expenditures:
  - They serve as a way to improve a firm's competitiveness in posterior periods.
- OPEC pricing, how to sustain collusion in the long run...

Games we will analyze:

1. Complete information:
    1. Simultaneous-move games with complete information.
    2. Sequential-move games with complete information.
2. Incomplete information:
    1. Simultaneous-move games with incomplete information.
    2. Sequential-move games with incomplete information.

Rules of a General Game (informal):

The rules of a game seek to answer the following questions:

1. Who is playing ?$\leftarrow$ set of players (I)
2. What are they playing with ?$\leftarrow$ Set of available actions for every player i (S_i)
3. Where each player gets to play ?$\leftarrow$ Order, or time structure of the game.
4. How much players can gain (or lose) ?$\leftarrow$ Payoffs (measured by a utility function $U_i(s_i, s_{-i})$

1. We assume **Common knowledge** about the rules of the game.

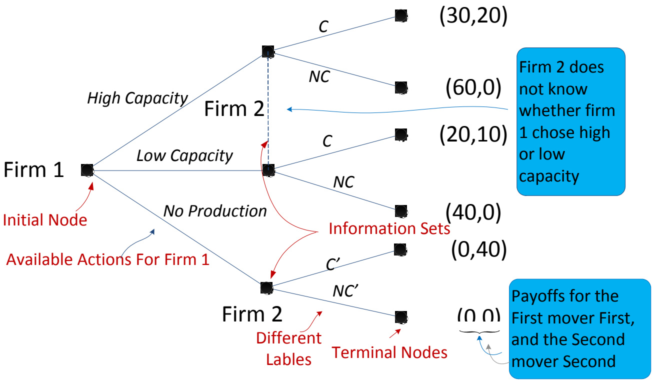
- As a player, I know the answer to the above four questions (rules of the game)
- In addition, I know that you know the rules, and...
- that you know that I know that you know the rules,.....(ad infinitum).

Two ways to graphically represent games

- Extensive form
    - We will use a game tree (next slide).
- Normal form (also referred as "strategic form").
    - We will use a matrix.

Example of a game tree

- Consider the following sequential-move game played by firms 1 and 2:
  - We will use a matrix



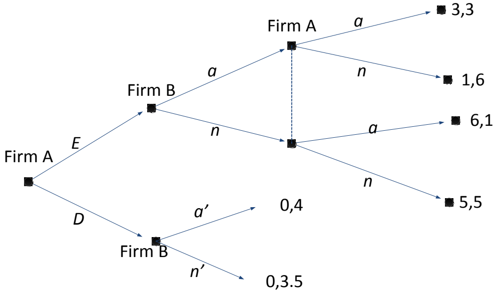
Information sets

- An information set is graphically represented with two or more nodes connected by a dashed line, (or a "sausage") including all these connected nodes.
- It represents that the player called to move at that information set cannot distinguish between the two or more actions chosen by his opponent before he is called to move.
- Hence, the set of available actions must be the same in all the nodes included on that information set.

Example

- **Lets practice how to depict a game tree of a strategic situation on an industry:**
- Firm A decides whether to enter firm B's industry. Firm B observes this decision.
  - If firm A stays out, firm B alone decides whether to advertise. In this case, firm A obtains zero profits, and firm B obtains $4 million if it advertises and $3.5 million if it does not.
  - If firm A enters, both firms simultaneously decide whether to advertise, obtaining the following payoffs.
    - If both advertise, both firms earn $3 million.
    - If none of them advertise, both firms earn $5 million.
    - If only one firm advertises, then it earns $6 million and the other firm earns $1 million.

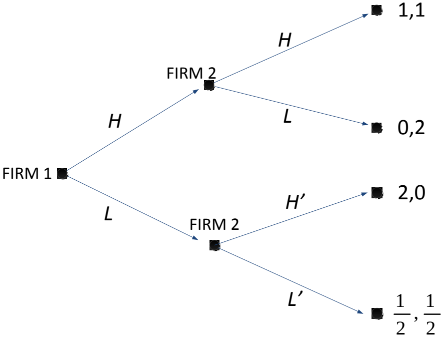
Example (continued)



- Let E and D denote firm A's initial alternatives of entering and not entering B's industry.
- Let a and n stand for "advertise" and "not advertise", respectively.
- Note that simultaneous advertising decisions are captured by firm A's information set.

Strategy: Definition of Strategy

- Lets practice finding the strategies of firm 1 and 2 in the following game tree:
  - We will use a matrix



Strategies for firm 1 : H and L.
Strategies for firm 2 : HH’; HL’; LH’; LL’

Strategy space and Strategy profile

- **Strategy space:** It is a set comprising each of the possible strategies of player i.
  - From our previous example:
    - $S_1 = \{H, L\}$ for firm 1
    - $S_2 = \{HH', HL', LH', LL'\}$ for firm 2.
- **Strategy profile**
  - It is a vector (or list) describing a particular strategy for every player in the game. For instance, in a two-player game

$$s = (s_1, s_2)$$

    where s_1 is a specific strategy for firm 1.(for instance, $s_1 = H$), and s_2 is a specific strategy for firm 2, e.g., $s_2 = LH'$.
  - More generally, for N players, a strategy profile is a vector with N components,

$$s = (s_1, s_2, s_3, ..., s_n)$$

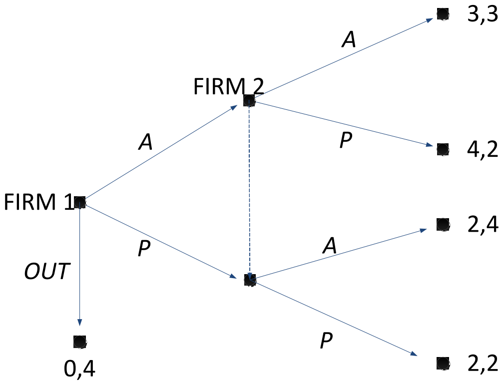
Strategy profile:

- In order to represent the strategies selected by all players except player i, we write:

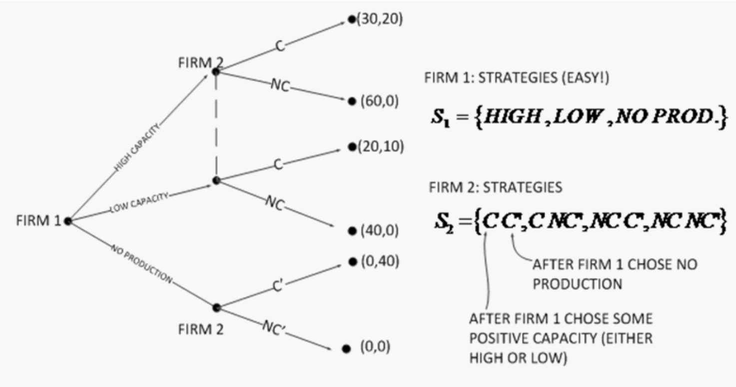
$$s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$$

  (Note that these strategies are potentially different)
- We can hence write, more compactly, as strategy profile with only two elements:
  The strategy player i selects, s_i, and the strategies chosen by everyone else, s_{-i}, as : $s = (s_i, s_{-i})$
- **Example:**
  - Consider a strategy profile s which states that player 1 selects B, player 2 chooses X, and player 3 selects Y, i.e., $s = (B, X, Y)$.Then,
    - $s_{-1} = (X, Y)$,
    - $s_{-2} = (B, Y)$, and
    - $s_{-3} = (B, X)$.

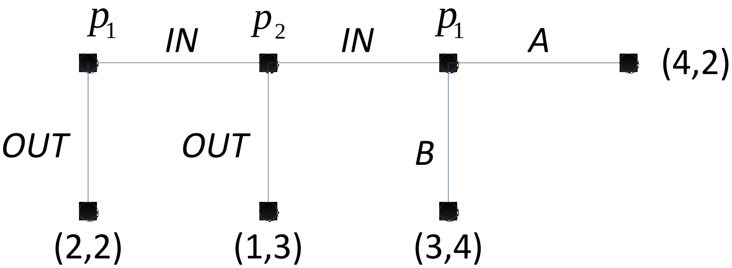
- Lets practice finding strategy sets in the following game tree:



- Let's define firm 1 and 2's available strategies in the first example of a game tree we described a few minutes ago:



Another example: The Centipede game



- Strategy set for player 2 : $S_2 = \{IN, OUT\}$
- Strategy set for player 1 : $S_1 = \{IN\ A, IN\ B, OUT\ A, OUT\ B\}$

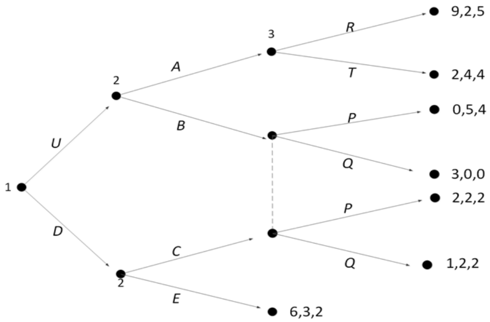
One second...

- Why do we have to specify my future actions after selecting "out" ? Two reasons:

1. Because of potential mistakes:
    - Imagine I ask you to act on my behalf, but I just inform you to select "out" at the initial node. However, you make a mistake (i.e., you play "In"), and player 2 responds with "In" as well. What would you do now??
    - With a strategy (complete contingent plan) you would know what to do even in events that are considered off the equillibrium path.
2. Because player 1's action later on affects player 2's actions, and ...
    - ultimately player 2's actions affects player 1's decision on whether to play "In" or "Out" at the beginning of the game.
    - This is related with the concept of backwards induction that we will discuss when solving sequential-move games.)

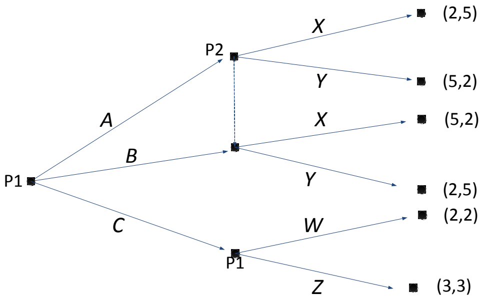
Some extensive-form games

- Let's now find the strategy spaces of a game with three players:



- $S_1 = \{U, D\}$
- $S_2 = \{AC, AE, BC, BE\}$; and
- $S_3 = \{RP, RQ, TP, TQ\}$

Some extensive-form games (Cont'd)



- S_1 = {AW,BW,CW,AZ,BZ,CZ}
- S_2 = {X,Y}

- **When a game is played simultaneously, we can represent it using a matrix**
  - *Example:* Prisoners' Dilemma game.

		Prisoner 2	
		Confess	Don't Confess
Prisoner 1	Confess	$-5, -5$	$0, -15$
	Don't Confess	$-15, 0$	$-1, -1$

Matrix (normal-form) representation

- A normal-form game includes three components
  - A **finite set of players** $N = \{1, 2, ..., n\}$;
  - A **collection of pure strategies** $\{S_1, S_2, ..., S_n\}$, i.e., a strategy space for every player i $S_i = \{s_{1i}, s_{2i}, ..., s_{1n}\}$; and
  - A set of payoff functions $\{v_1, v_2, ..., v_n\}$, each assigning a payoff value to each strategy profile, that is,

$$v_i : S_1 \times S_2 \times ... \times S_n \rightarrow \mathbb{R} \quad \text{for every player } i \in N$$

- **Yet, another example of a simultaneous-move game**
  - Pareto-coordination game.

		Firm 2	
		Superior tech.	Inferior tech.
Firm 1	Superior tech.	2, 2	0, 0
	Inferior tech.	0, 0	1, 1

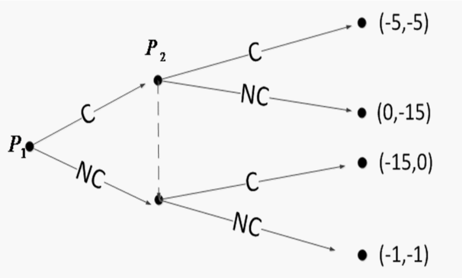
- **Yet, another example of a simultaneous-move game**
  - The game of "chicken."

		Dean	
		Straight	Swerve
James	Straight	0, 0	3, 1
	Swerve	1, 3	2, 2

Normal (Strategic) Form

- We can alternatively represent simultaneous-move games using a game tree, as long as we illustrate that players choose their actions without observing each others' moves, i.e., using information sets, as we do next for the prisoner's dilemma game:
- Extensive form representation of the Prisoner's Dilemma game :

		P_2	
		C	NC
P_1	C	-5,-5	0,-15
	NC	-15,0	-1,-1



- **We are done discribing games!!**
  - We will return to some additional properties of game trees later on, but only for a second.
- **Let's start solving games!!**
  - We will use solution concepts that will help us predict the precise strategy that every player selects in the game.
- **Our goal:**
  - To be as precise as possible in our equilibrium predictions.
  - Hence, we will present (and rank) solution concepts in terms of their predictive power.

Evaluating solution concepts

- Existence
- Uniqueness (or small set of solutions)
- Robustness to small perturbations.

Evaluating equilibrium outcomes

- **Pareto dominance:**
  - A strategy profile $s \in S$ Pareto dominates strategy profile $s' \in S$ if

$$\begin{aligned} v_i(s) &\geq v_i(s') \quad \text{for all } i \in N, \text{ and} \\ v_i(s) &> v_i(s') \quad \text{for at least one individual } i \in N \end{aligned}$$

    In this case we say that s' is Pareto dominated by s.
  - A strategy profile $s \in S$ is Pareto optimal if it is not Pareto dominated by any other strategy profile.