Revenue Equivalence Theorem

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- So far, several different auction types have been considered.
- The question remains: How does the expected revenue for the seller vary across the different types of auction?
  - Vickrey (1961) and Myerson (1981) were about to prove what is now known as the *revenue equivalence theorem*, stating that under certain, general conditions, all auction types produce the same expected revenue.

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- Four conditions for revenue equivalence
  1. Each bidder's type is drawn from a "well behaved" distribution. (The CDF must be strictly increasing and continuous)
  2. Bidders are risk neutral.
  3. The bidder with the highest type wins.
  4. The bidder with the lowest possible type ($\underline{\theta}$) has an expected payoff of zero.

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- Before we start an example, a bit of background.
- Imagine that we are drawing a series of independent realizations from some distribution $F(.)$. Given the sample of draws $(x_1, x_2, ..., x_n)$, we can consider the ranking of the realized values and rank them from highest to lowest.
- The highest draw would be $x_n^{[1]} = \max\{x_1, x_2, ..., x_n\}$, also known as the first-order statistic.
- Similarly, the second highest draw, $x_n^{[2]}$ would be the second order statistic, and so on.

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- Let's see if the expected revenue from a first-price and second-price auction are equivalent.
- Starting with the first-price auction, we'll assume that there are n symmetric bidders whose types are drawn from a uniform $[0, 1]$ distribution.
- All bidders use the same bidding strategy $s(\theta)$. This implies that the revenue received by the seller would be the first-order statistic of some *bidding* distribution $G(.)$.
- Hence, the expected revenue of the seller is $E(b_n^{[1]})$, which is the expected value of the first-order statistic of the bid function.

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- As previously shown, the symmetric Bayesian Nash equilibrium with n bidders whose types are uniformly distributed over an interval $[\underline{\theta}, \bar{\theta}]$ is given by $s(\theta) = \theta\left(\frac{n-1}{n}\right)$ (Bidders all bid a share $\frac{n-1}{n}$ of their actual valuation, θ).
- The winning bid will be

$$\max\left\{\frac{\theta_1(n-1)}{n}, \frac{\theta_2(n-1)}{n}, \ldots, \frac{\theta_n(n-1)}{n}\right\},$$

which is just the first-order statistic from n draws on the uniform distribution $[0, \frac{n-1}{n}]$.

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- We then need to compare:
  - **FPA**: The expected revenue from the first-order statistic of the bidding function, i.e., the first-order statistic from n draws on the uniform distribution $[0, \frac{n-1}{n}]$, against
  - **SPA**: The expected revenue from the second-order statistic of the value function, i.e., the second-order statistic from n draws on the uniform distribution $[0, 1]$.

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- We'll need to derive the CDF of the first-order statistic for the uniform distribution in order to calculate the expected value.
- $F_n^{[1]}(x)$ is equal to the probability that all the n bids that were drawn are less than or equal to x, and hence

$$F_n^{[1]}(x) = \Pr\{\max\{b_1, b_2, ..., b_n\} \leq x\} = (F(x))^n$$

  - Intuitively, the last equality follows from the fact that n values are drawn independently.
- For the uniform distribution, the bid of each agent b_i is uniformly drawn from $[0, \frac{n-1}{n}]$, so that $F(b_i) = \frac{n}{n-1} b_i$.

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- Thus, the distribution of the first-order statistic is

$$F_n^{[1]}(x) = [F(x)]^n = \left(\frac{nx}{n-1}\right)^n \quad \text{for} \quad 0 \leq x \leq \frac{n-1}{n}$$

- We can obtain the PDF by differentiating with respect to x

$$f_n^{[1]}(x) = n\frac{n}{n-1}\left(\frac{nx}{n-1}\right)^{n-1} = \frac{n}{x}\left(\frac{nx}{n-1}\right)^n$$

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- Lastly, we can calculate the expected first-order statistic from Bayesian Nash equilibrium bids in a first-price auction

$$
\begin{aligned}
E(b_n^{[1]}) &= \int_0^{\frac{n-1}{n}} x f_n^{[1]}(x)dx = \int_0^{\frac{n-1}{n}} n\left(\frac{nx}{n-1}\right)^n dx \\
&= \left[\frac{n}{n+1}\left(\frac{n}{n-1}\right)^n x^{n+1}\right|_0^{\frac{n-1}{n}} = \frac{n-1}{n+1}
\end{aligned}
$$

- Thus, the seller's expected revenue from the first-price sealed-bid auction with n bidders whose valuations are independently drawn from the uniform distribution over $[0,1]$ is equal to $\frac{n-1}{n+1}$.

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- Now let's do the same thing for a second-price auction.
- In this case, all bidders will bid their valuation, and thus the expected revenue will be equal to the expected second-order statistic of a series of n draws from the uniform distribution $[0, 1]$ (Since the winner pays the second highest bid).
- Let's derive the distribution again.

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- The CDF follows $F_n^{[2]}(x) = \Pr\{x_n^{[2]} \leq x\}$.
    - The event $\{x_n^{[2]} \leq x\}$ can occur in one of two distinct (mutually exclusive) ways.
- In the first event, all the draws x_i are below x, or $x_i \leq x$ for all $i = 1, 2, ..., n$.
- In the second event, for $n - 1$ of the draws, $x_i \leq x$ and for only one draw j, $x_j > x$.
    - This event can occur in n different ways:
    - (1) $x_1 > x$ and for all $i \neq 1$, $x_i \leq x$,
    - (2) $x_2 > x$ and for all $i \neq 2$, $x_i \leq x$, and so on up to n.

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- We can therefore define the CDF as follows

$$
\begin{aligned}
F_n^{[2]}(x) &= \overbrace{\Pr\{\max\{b_1, b_2, ..., b_n\} \leq x\}}^{\text{First Event}} \\
&\quad + \underbrace{\sum_{i=1}^{n} \Pr\{x_i > x \text{ and for all } j \neq i,\, x_j \leq x\}}_{\text{Second Event}} \\
&= [F(x)]^n + \sum_{i=1}^{n} (1 - F(x))[F(x)]^{n-1}
\end{aligned}
$$

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- Rearranging

$$\begin{aligned} F_n^{[2]}(x) &= [F(x)]^n + n(1 - F(x))[F(x)]^{n-1} \\ &= n[F(x)]^{n-1} - (n-1)[F(x)]^n \end{aligned}$$

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- The bid of each agent b_i in the second-price sealed-bid auction is his valuation, which we assumed is uniformly drawn from $[0, 1]$, so that $F(b_i) = b_i$, and we obtain

$$\begin{aligned} F_n^{[2]}(x) &= n[F(x)]^{n-1} - (n-1)[F(x)]^n \\ &= nx^{n-1} - (n-1)x^n \end{aligned}$$

- Like before, we obtain the PDF by differentiating with respect to x, obtaining

$$f_n^{[2]}(x) = n(n-1)x^{n-2} - n(n-1)x^{n-1}$$

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- Lastly, the expected second-order statistic is

$$
\begin{aligned}
E(\theta_n^{[2]}) &= \int_0^1 x f_n^{[2]} dx = \int_0^1 [n(n-1)x^{n-1} - n(n-1)x^n] dx \\
&= \left[(n-1)x^n - \frac{n(n-1)x^{n+1}}{n+1} \right|_0^1 = \\
&= (n-1) - \frac{n(n-1)}{n+1} = \frac{n-1}{n+1}
\end{aligned}
$$

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- Thus, the expected revenue for the seller is the same in either a first or second-price sealed bid auction.
  - Books by Krishna (2002) and Milgrom (2004) both explore this topic in greater detail.
- You can try the same process with other types of auction.