EconS 501 - Micro Theory I
Recitation #8b - Uncertainty II

1. **Exercise 6.E.1:** The purpose of this exercise is to show that preferences may not be transitive in the presence of regret. Let there be S states of the world, indexed by $s = 1, ..., S$. Assume that state s occurs with probability π_s. Define the expected regret associated with lottery $x = (x_1, ..., x_s)$ relative to lottery $x' = \left(x_1^{'}, ..., x_s^{'}\right)$ by

$$\sum_{s=1}^{S} \pi_s h\left(\max\left\{0, x_s^{'} - x_s\right\}\right),$$

where $h(\cdot)$ is a given increasing function. [We call $h(\cdot)$ the *regret valuation function*; it measures the regret the individual has after the state of nature is known.] We define x to be at least as good as x' in the presence of regret if and only if the expected regret associated with x relative to x' is not greater than the expected regret associated with x' relative to x.

Suppose that $S = 3, \pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$, and $h(x) = \sqrt{x}$. Consider the following three lotteries:

$$\begin{aligned} x &= (0, -2, 1), \\ x' &= (0, 2, -2), \\ x'' &= (2, -3, -1). \end{aligned}$$

Show that the preference ordering over these three lotteries is not transitive.

- **Answer:** Denote by $R(x, x')$ the expected regret associated with lottery x relative to x', and similarly for the other lotteries. A direct calculation yields:

$$\begin{aligned} R(x, x') &= \frac{2}{3} \simeq 0.66, \quad \text{and} \quad R(x', x) = \frac{\sqrt{3}}{3} \simeq 0.577, \\ R(x', x'') &= \frac{\left(\sqrt{2}+1\right)}{3} \simeq 0.804, \quad \text{and} \quad R(x'', x') = \frac{\sqrt{5}}{3} \simeq 0.745, \\ R(x'', x) &= \frac{\left(\sqrt{2}+1\right)}{3} \simeq 0.804, \quad \text{and} \quad R(x, x'') = \frac{\sqrt{2}}{3} \simeq 0.471. \end{aligned}$$

Hence,

$$\begin{aligned} R(x, x') &> R(x', x), \\ R(x', x'') &> R(x'', x'), \\ R(x'', x) &> R(x, x''). \end{aligned}$$

Thus, x' is preferred to x, x'' is preferred to x', but x is preferred to x''.

2. **Exercise 6.F.2:** The purpose of this exercise is to explain the outcomes of the experiments described in Example 6.F.1 (page 207 MWG) by means of the theory of *nonunique prior beliefs* of Gilboa and Schmeidler (1989).

We consider a decision maker with a Bernoulli utility function $u(\cdot)$ defined on $\{0, 1000\}$. We normalize $u(\cdot)$ so that $u(0) = 0$ and $u(1000) = 1$.
The probabilistic belief that the decision maker might have on the color of the H-ball being white is a number $\pi \in [0, 1]$. We assume that the decision maker has, not a single belief but a *set* of beliefs given by a subset P of $[0, 1]$. The actions that he may take are denoted R or H with R meaning that he chooses the R-ball and H meaning that he chooses the H-ball.
As in Example 6.F.1, the decision maker is faced with two different choice situations. In choice situation W, he receives 1000 dollars if the ball chosen is white and 0 dollars otherwise. In choice situation B, he receives 1000 dollars if the ball chosen is black and 0 dollars otherwise.
For each of the two choice situations, define his utility function over the actions R and H in the following way:
For situation $W, U_W : \{R, H\} \to \mathbb{R}$ is defined by

$$U_W(R) = .49 \text{ and } U_W(H) = \min\{\pi : \pi \in P\}.$$

For situation B, $U_B : \{R, H\} \to \mathbb{R}$ is defined by

$$U_B(R) = .51 \text{ and } U_B(H) = \min\{(1 - \pi) : \pi \in P\}.$$

Namely, his utility from choice R is the expected utility of 1000 dollars with the (objective) probability calculated from the number of white and black balls in urn R. However, his utility from choice H is the expected utility of 1000 dollars with the probability associated with the most pessimistic belief in P.

a. Prove that if P consists of only one belief, then U_W and U_B are derived from a von Neumann-Morgenstern utility function and that $U_W(R) > U_W(H)$ if and only if $U_B(R) < U_B(H)$.

- **Answer:** If $P = \{\pi\}$, then $U_W(H) = \pi$ and $U_B(H) = 1 - \pi$. Hence they are determined from the expected utility $\pi u(1000) + (1 - \pi) u(0)$. Moreover, $U_W(R) > U_W(H)$ if and only if $0.49 > \pi$. But this is equivalent to $0.51 < 1 - \pi$, which is, in turn, equivalent to $U_B(R) < U_B(H)$.

b. Find a set P for which $U_W(R) > U_W(H)$ and $U_B(R) > U_B(H)$.

- **Answer:** We have $U_W(R) > U_W(H)$ if and only if $0.49 > \min P$. We have $U_B(R) > U_B(H)$ if and only if $0.51 > \min\{1 - \pi : \pi \in P\}$, which is equivalent to $0.49 < \max P$. Hence $\min P < 0.49 < \max P$ if and only if $U_W(R) > U_W(H)$ and $U_B(R) > U_B(H)$.

3. **Exercise 11.6:** Esperanza has been an expected utility maximizer ever since she was five years old. As a result of the strict education she received at an obscure British boarding school, her utility function u is strictly increasing and strictly concave. Now, at the age of thirty-something, Esperanza is evaluating an asset with stochastic outcome R which is normally distributed with mean μ and variance σ^2. Thus, its density function is given by

$$f(r) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2\right\}.$$

(a) Show that Esperanza's expected utility from R is a function of μ and σ^2 alone. Thus, show that

$$E[u(R)] = \phi\left(\mu, \sigma^2\right).$$

- **Answer (a):** Note that

$$E[u(R)] = \int_{-\infty}^{\infty} u(s) \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2\right\} ds = \phi\left(\mu, \sigma^2\right).$$

(b) Show that $\phi(\cdot)$ is increasing in μ.

- **Answer (b):** [See the figure below] Normalize $u(\cdot)$ such that $u(\mu) = 0$. Differentiating, we have

$$\frac{\partial E[u(R)]}{\partial \mu} = \frac{1}{\sigma^2}\int_{-\infty}^{\infty} u(s)(s-\mu) f(s)\, ds > 0,$$

since the terms $[u(s)(s-\mu)]$ and $f(s)$ are positive for all s.

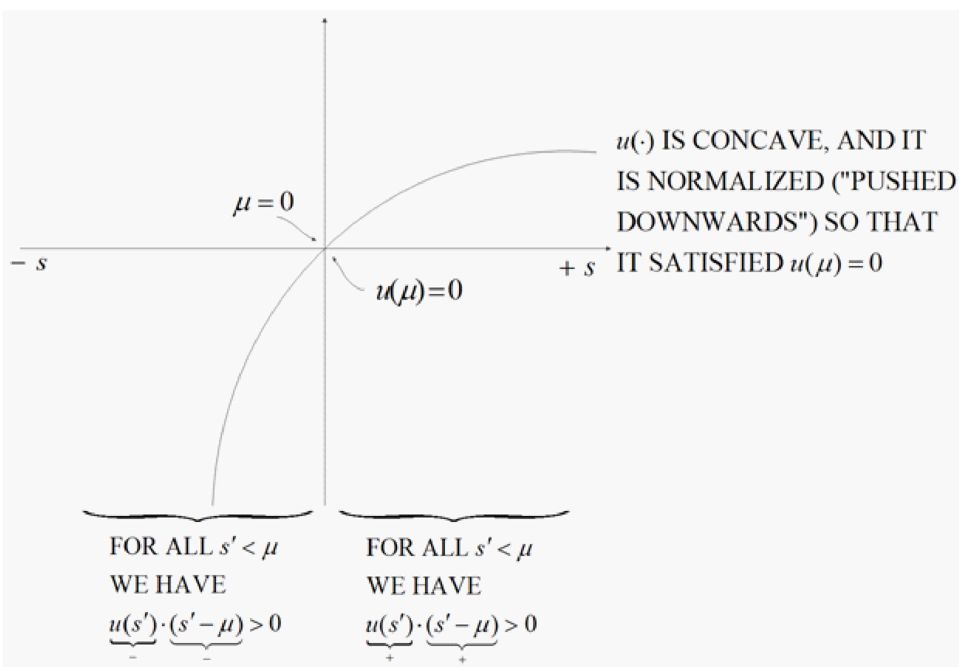

Figure 1

(c) Show that $\phi(\cdot)$ is decreasing in σ^2.

- **Answer (c):** [See the figure at the end of this handout] Now we have

$$
\begin{aligned}
\frac{\partial E[u(R)]}{\partial \sigma^2} &= \frac{1}{\sigma^3}\int_{-\infty}^{\infty} u(s)\left((s-\mu)^2-\sigma^2\right) f(s)\,ds \\
&< \frac{1}{\sigma^3}\int_{-\infty}^{\infty} u'(\mu)(s-\mu)\left((s-\mu)^2-\sigma^2\right) f(s)\,ds \\
&= \frac{u'(\mu)}{\sigma^3}\left\{\int_{-\infty}^{\infty}(s-\mu)^3 f(s)\,ds-\sigma^2\int_{-\infty}^{\infty}(s-\mu) f(s)\,ds\right\} \\
&= 0.
\end{aligned}
$$

The first inequality follows from the concavity of $u(\cdot)$ and the normalization imposed; the last equality follows from the fact that R is normally distributed and, hence $E\left[R-E[R]^k\right]=0$ for k odd.

4. **Exercise 6.C.7:** Prove that, in Proposition 6.C.2, condition (iii) implies condition (iv), and (iv) implies (i).

(i) $r_A(x, u_2) \geq r_A(x, u_1)$ for every x.

(iii) $c(F, u_2) \leq c(F, u_1)$ for any $F(\cdot)$.

(iv) $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$ for any x and ε.

- Suppose first that condition (iii) holds. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Denote by $F(\cdot)$ the distribution function that puts probability $\frac{1}{2} - \pi(x, \varepsilon, u_2)$ on $x - \varepsilon$ and $\frac{1}{2} + \pi(x, \varepsilon, u_2)$ on $x + \varepsilon$. That is,

$$F(z) = \begin{cases} 0 & \text{if } z < x - \varepsilon, \\ \frac{1}{2} - \pi(x, \varepsilon, u_2) & \text{if } x - \varepsilon \leq z < x + \varepsilon, \\ 1 & \text{if } x + \varepsilon \geq z. \end{cases}$$

then $c(F, u_2) = x$. By (iii), $c(F, u_1) \geq x$. Thus $u_1(c(F, u_1)) \geq u_1(x)$. But by definition 6.C.2 (on page 186 MWG) we have

$$\begin{aligned} & u_1(c(F, u_1)) \\ = & \left(\frac{1}{2} - \pi(x, \varepsilon, u_2)\right) u_1(x - \varepsilon) + \left(\frac{1}{2} + \pi(x, \varepsilon, u_2)\right) u_1(x + \varepsilon) \\ = & \left(\frac{1}{2}\right) u_1(x - \varepsilon) + \left(\frac{1}{2}\right) u_1(x + \varepsilon) + \pi(x, \varepsilon, u_2)(u_1(x + \varepsilon) - u_1(x - \varepsilon)) \end{aligned}$$

and

$$\begin{aligned} & u_1(x) \\ = & \left(\frac{1}{2} - \pi(x, \varepsilon, u_1)\right) u_1(x - \varepsilon) + \left(\frac{1}{2} + \pi(x, \varepsilon, u_1)\right) u_1(x + \varepsilon) \\ = & \left(\frac{1}{2}\right) u_1(x - \varepsilon) + \left(\frac{1}{2}\right) u_1(x + \varepsilon) + \pi(x, \varepsilon, u_1)(u_1(x + \varepsilon) - u_1(x - \varepsilon)). \end{aligned}$$

Thus the last inequality is equivalent to $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$. Hence, condition (iv) holds.

- Suppose now that condition (iv) holds. Since

$$\begin{aligned} \pi(x, 0, u_1) &= \pi(x, 0, u_2) = 0, \\ \pi(x, 0, u_2) &\geq \pi(x, 0, u_1) = 0, \end{aligned}$$

(iv) implies that

$$\frac{\partial \pi(x, 0, u_2)}{\partial \varepsilon} \geq \frac{\partial \pi(x, 0, u_2)}{\partial \varepsilon}.$$

Since

$$r_A(x, u_1) = \frac{4\partial \pi(x, 0, u_1)}{\partial \varepsilon} \quad \text{and} \quad r_A(x, u_2) = \frac{4\partial \pi(x, 0, u_2)}{\partial \varepsilon},$$

(i) follows.

5. **Exercise 6.C.8:** Assume that the Bernoulli utility function $u(.)$ exhibits decreasing absolute risk aversion. Show that for the asset demand model of Example 6.C.2 (page 188 MWG), the optimal allocation between the safe and the risky assets implies that the allocation of wealth on the risky asset is increasing as w rises (i.e., the risky asset is a normal good).

   - **Answer:** Let w_1 and w_2 be two wealth levels such that $w_1 > w_2$ and define $u_1(z) = u(w_1 + z)$ and $u_2(z) = u(w_2 + z)$, then $u_2(\cdot)$ is a concave transformation of $u_1(\cdot)$ by (i) and (ii) of Proposition 6.C.3 on page 191 in MWG. It was shown in Example 6.C.2 (continued) that the demand for the risky asset of $u_1(\cdot)$ is greater than that of $u_2(\cdot)$. This means that the demand for the risky asset of $u(\cdot)$ is greater at wealth level w_1 than at w_2. Intuitively, if the demand for risky assets is larger for the individual with the less concave utility function, $u_1(\cdot)$, then if we evaluate $u_1(\cdot)$ at a higher wealth level w_1 than the wealth level at which we evaluate $u_2(\cdot)$, the ranking between the risky assets of individual 1 and 2 still holds.

6. **Exercise 6.C.15:** Assume that, in a world with uncertainty, there are two assets. The first is a riskless asset that pays 1 dollar. The second pays amounts a and b with probabilities of π and $1 - \pi$, respectively. Denote the demand for the two assets by $(x_1, x_2,)$.

Suppose that a decision maker's preferences satisfy the axioms of expected utility theory and that he is a risk averter. The decision maker's wealth is 1, and so are the prices of the assets. Therefore, the decision maker's budget constraint is given by

$$x_1 + x_2 = 1, \quad x_1, x_2 \in [0, 1].$$

- Throughout this answer, we assume that $a \neq b$, because, otherwise, there would be no uncertainty involved in the payment of the second asset.

a. Give a simple *necessary condition* (involving a and b only) for the demand for the riskless asset to be strictly positive.

   - **Answer:** If $\min\{a, b\} \geq 1$, the risky asset pays at least as high a return as the riskless asset at both states, and a strictly higher return at one of them. Then all the wealth is invested to the risky asset. Thus, $\min\{a, b\} < 1$ is a necessary condition for the demand for the riskless asset to be strictly positive.

b. Give a simple *necessary condition* (involving a, b, and π only) for the demand for the risky asset to be strictly positive.

   - **Answer:** If $\pi a + (1 - \pi) b \leq 1$, then the expected return does not exceed the payments of the riskless asset and hence the risk-averse decision maker does not demand the risky asset at all. Thus $\pi a + (1 - \pi) b > 1$ is a necessary condition for the demand for the risky asset to be strictly positive.

In the next three parts, assume that the conditions obtained in (a) and (b) are satisfied. In the following answers, we assume that the demands for both assets are always positive.

c. Write down the first-order conditions for utility maximization in this asset demand problem.

- **Answer:** Since the prices of the two assets are equal to one, their marginal utilities must be equal. Thus

  $$\pi u'(x_1 + x_2 a) + (1-\pi) u'(x_1 + x_2 b) = \pi a u'(x_1 + x_2 a) + (1-\pi) b u'(x_1 + x_2 b).$$

  That is,

  $$\pi (1-a) u'(x_1 + x_2 a) + (1-\pi)(1-b) u'(x_1 + x_2 b) = 0.$$

  This and $x_1 + x_2 = 1$ constitute the first-order condition.

d. Assume that $a < 1$. Show by analyzing the first-order conditions that $\frac{dx_1}{da} \leq 0$.

- **Answer:** Taking b as constant, define

  $$\phi(a, \pi, x_1) = \pi (1-a) u'(x_1 + (1-x_1) a) + (1-\pi)(1-b) u'(x_1 + (1-x_1) b),$$

  then

  $$\begin{aligned} \frac{\partial \phi}{\partial a} &= -\pi u'(x_1 + (1-x_1) a) + \pi (1-a)(1-x_1) u''(x_1 + (1-x_1) a) < 0, \\ \frac{\partial \phi}{\partial x_1} &= \pi (1-a)^2 u''(x_1 + (1-x_1) a) + (1-\pi)(1-b)^2 u''(x_1 + (1-x_1) b) < 0. \end{aligned}$$

  Thus, by the implicit function theorem (Theorem M.E.1),

  $$\frac{dx_1}{da} = -\frac{\frac{\partial \phi}{\partial a}}{\frac{\partial \phi}{\partial x_1}} < 0.$$

e. Which sign do you conjecture for $\frac{dx_1}{d\pi}$? Give an economic interpretation.

- **Answer:** It follows from the condition of (b) that $b > 1$, that is, that a is the worse outcome of the risky asset. Thus, if the probability π of the worse outcome is increased, then it is anticipated that the demand for the riskless asset is increased.

f. Can you prove your conjecture in (e) by analyzing the first-order conditions?

- **Answer:** Since $b > 1$,

  $$\begin{aligned} \frac{\partial \phi}{\partial \pi} &= (1-a) u'(x_1 + (1-x_1) a) - (1-b) u'(x_1 + (1-x_1) b) \\ &= (1-a) u'(x_1 + (1-x_1) a) + (b-1) u'(x_1 + (1-x_1) b) > 0, \end{aligned}$$

  because $a < 1 < b$. Thus

  $$\frac{dx_1}{d\pi} = -\frac{\frac{\partial \phi}{\partial \pi}}{\frac{\partial \phi}{\partial x_1}} > 0,$$

  as anticipated.