EconS 501 - Micro Theory I[1]
Recitation #7 - Competitive Markets

Exercise 1

1. **Exercise 12.5, NS:** Suppose that the demand for stilts is given by $Q = 1,500 - 50P$ and that the long-run total operating costs of each stilt-making firm in a competitive industry are given by $C(q) = 0.5q^2 - 10q$.

Entrepreneurial talent for stilt making is scarce. The supply curve for entrepreneurs is given by$Q_s = 0.25w$ where w is the annual wage paid. Suppose also that each stilt-making firm requires one (and only one) entrepreneur (hence, the quantity of entrepreneurs hired is equal to the number of firms). Long-run total costs for each firm are then given by:

$$C(q, w) = 0.5q^2 - 10q + w$$

(a) What is the long-run equilibrium quantity of stilts produced? How many stilts are produced by each firm? What is the long-run equilibrium price of stilts? How many firms will there be? How many entrepreneurs will be hired, and what is their wage?

(b) Suppose that the demand for stilts shifts outward to

$$Q = 2,428 - 50P$$

How would you know answer the questions posed in part a.

(c) Because stilt-making entrepreneurs are the cause of the upward-sloping long-run supply curve in this problem, they will receive all rents generated as industry output expands. Calculate the increase in rents between parts (a) and (b). Show that this value is identical to the change in long-run producer surplus as measured along the stilt supply curve.

Solution:

This problem introduces the concept of increasing input costs into long-run analysis by assuming that entrepreneurial wages are bid up as the industry expands. Solving part (b) can be a bit tricky; perhaps an educated guess is the best way to proceed.

(a) The equilibrium in the entrepreneur market requires $Q_s = 0.25w = Q_D = n$ or, solving for w, we obtain $w = 4n$.

Hence, given $C(q, w) = 0.5q^2 - 10q + w$, the marginal cost is $MC = q - 10$ and the average cost is $AC = 0.5q - 10 + \frac{w}{q} = 0.5q - 10 + \frac{4n}{q}$

In long run equilibrium the $MC = AC$, thus:

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$$q - 10 = 0.5q - 10 + \frac{4n}{q}$$

$$q - 0.5q = \frac{4n}{q}$$
$$0.5q = \frac{4n}{q}$$
$$q^2 = 8n \text{ then } q = \sqrt{8n}$$

Total output is given in terms of the number of firms by

$$Q = nq = n\sqrt{8n}$$

Now in terms of profit-maximization in perfectly competitive markets requires $P = MC$, or $q = P + 10$

Hence aggregate supply becomes $Q_S = nq = n(P + 10)$

Since total output requires $Q = n\sqrt{8n}$ and aggregate supply requires $Q = n(P + 10)$, we have

$$n\sqrt{8n} = n(P + 10)$$

$$P = \sqrt{8n} - 10$$

Therefore, we can use the demand function Q_D evaluated at the equilibrium price $P = \sqrt{8n} - 10$,

$$Q_D = 1500 - 50P = 1500 - 50(\sqrt{8n} - 10) = 2000 - 50\sqrt{8n}$$

Then since in equilibrium we require that demand is equal to supply, $Q_D = Q_S$, we obtain

$$2000 - 50\sqrt{8n} = n\sqrt{8n} \text{ thus } n = 50 \text{ entrepreneurs}$$

Finally, we can also calculate:

$$\text{Total output: } Q = n\sqrt{8n} = 1000$$

$$\text{Individual output by every firm: } q = \frac{Q}{n} = 20$$

$$\text{Equilibrium price: } P = q - 10 = 10$$

$$w = 4n = 200.$$

(b) Using the results of the previous part and if the demand function is $Q = 2,428 - 50P$ then,

$$(n\sqrt{8n}) = 2,428 + 50(\sqrt{8n} - 10)$$

$$(n\sqrt{8n}) + 50\sqrt{8n} = 2,928$$

$$(n + 50)\sqrt{8n} = 2,928, \text{ therefore } n = 72$$

and, we can then recalculate:

$$Q = n\sqrt{8n} = 1728$$

$$q = \frac{Q}{n} = 24$$

$$P = q - 10 = 14$$

$$w = 4n = 288.$$

So, as the demand shifts outward, the number of firms in the industry increases, the total production and firm production increases, the price increases and the wages increase.

(c) The long-run supply curve is upward sloping because as new firms enter the industry the cost curves shift up:

$$AC = 0.5q - 10 + \frac{4n}{q}$$

as n increases the average cost also increases.

Using linear approximations, the increase in the producers surplus (PS) from the supply curve is given by $4 \cdot 1000 + 0.5 \cdot 728 \cdot 4 = 5456$. If we use instead the supply curve for entrepreneurs the area is $88 \cdot 50 + 0.5 \cdot 88 \cdot 22 = 5368$. These two numbers agree roughly. To get exact agreement would require recognizing that the long-run supply curve here is not linear – it is slightly concave.

Exercise 2

2. **Exercise 12.9, NS:** Given an ad valorem tax rate (ad valorem tax is a tax on the value of transaction or a proportional tax on price) of t ($t = 0.05$ for a 5% tax), the gap between the price demanders pay and what suppliers receive is given by $P_D = (1+t)P_S$.

   (a) Show that, for an ad valorem tax,
   $\frac{d \ln P_D}{dt} = \frac{e_S}{e_S - e_D}$ and $\frac{d \ln P_S}{dt} = \frac{e_D}{e_S - e_D}$

   (b) Show that the excess burden of a small tax is
   $DW = -0.5 \frac{e_D e_S}{e_S - e_D} t^2 P_0 Q_0$

   (c) Compare these results to those for the case of a unit tax.

Solution:

This problem shows that the comparative statics results for ad valorem taxes are very similar to the results for per-unit taxes

(a) Given that the gap between the price demanders pay and what suppliers receive is $P_D = (1+t)P_S$

Then, the introduction of a tax implies a small price change, i.e., $dP_D = (1+t)dP_S + dt \cdot P_S = (1+t)dP_S + dt \cdot P_S$, where we can evaluate this expression at $t = 0$ since the tax is impossed before any tax was present. Hence, the previous expression collapses to $dP_D = dP_S + dt \cdot P_S =$.

We know also that

$$e_D = \frac{\partial Q_D}{\partial P} \cdot \frac{P}{Q_D} \text{ and } e_S = \frac{\partial Q_S}{\partial P} \cdot \frac{P}{Q_S}$$

In equilibrium with a tax rate of t, we will have

$$Q_D(P_D) = Q_S(P_S)$$

$$\frac{\partial Q_D}{\partial P_D} dP_D = \frac{\partial Q_S}{\partial P_S} dP_S$$

but since $dP_D = dP_S + dtP_S$ then,

$$\frac{\partial Q_D}{\partial P_D}(dP_S + dtP_S) = \frac{\partial Q_S}{\partial P_S} dP_S$$

$$\frac{\partial Q_D}{\partial P_D} dP_S + \frac{\partial Q_D}{\partial P_D} dtP_S = \frac{\partial Q_S}{\partial P_S} dP_S$$

rearranging

$$\frac{\partial Q_D}{\partial P_D} dtP_S = \frac{\partial Q_S}{\partial P_S} dP_S - \frac{\partial Q_D}{\partial P_D} dP_S$$

$$\frac{\partial Q_D}{\partial P_D} dtP_S = \left(\frac{\partial Q_S}{\partial P_S} - \frac{\partial Q_D}{\partial P_D} \right) dP_S$$

$$\frac{\frac{\partial Q_D}{\partial P_D}}{\frac{\partial Q_S}{\partial P_S} - \frac{\partial Q_D}{\partial P_D}} = \frac{dP_S}{dt} \cdot \frac{1}{P_S} \ (*)$$

Hence, we can express this ratio in terms of elasticities, as follows,

$$\frac{dP_S}{dt}\cdot\frac{1}{P_S}=\frac{d\ln P_S}{dt}=\frac{\frac{\partial Q_D}{\partial P_D}}{\frac{\partial Q_S}{\partial P_S}-\frac{\partial Q_D}{\partial P_D}}=\frac{\frac{\partial Q_D}{\partial P_D}\cdot\frac{P_0}{Q}}{\frac{\partial Q_S}{\partial P_S}\cdot\frac{P_0}{Q}-\frac{\partial Q_D}{\partial P_D}\cdot\frac{P_0}{Q}}=\frac{e_D}{e_S-e_D}$$

(b) A linear approximation of the DWL accompanying a small tax dt is given by:

$$DWL=0.5(P_0dt)(dQ)$$

as depicted in the next figure.

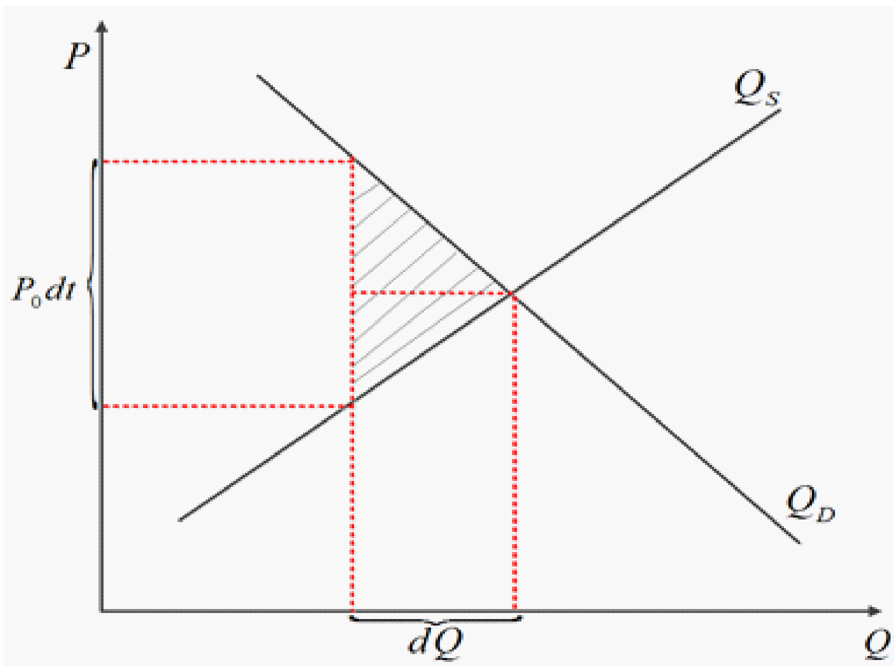

Figure 1. Introducing a tax in a competitive equilibrium market.

Since $e_D=\frac{\partial Q_D}{\partial P}\cdot\frac{P}{Q_D}=\frac{\mathrm{dln}Q}{\mathrm{dln}P}$

then $dQ=e_D\frac{Q_0}{P}dP$ and substituting into DWL

$$DWL=0.5(P_0dt)\left(e_D\frac{Q_0}{P}dP\right)$$
$$DWL=0.5\cdot P_0\cdot(dt)^2\cdot e_D\cdot Q_0\cdot\frac{dP}{dt}\frac{1}{P}$$

but from (*) we know $\frac{dP_D}{dt}\cdot\frac{1}{P_D}=\frac{e_S}{e_D-e_S}$ then

$$DWL=0.5\cdot P_0\cdot(dt)^2\cdot e_D\cdot Q_0\cdot\frac{e_S}{e_D-e_S}$$
$$DWL=0.5\left(\frac{e_De_S}{e_D-e_S}\right)(dt)^2Q_0P_0$$

We can now generalize this result for any small t:

$$DWL = 0.5 \left(\frac{e_D e_S}{e_D - e_S} \right) t^2 Q_0 P_0$$

(c) The unit tax described in this chapter is equivalent to the value of the ad-valorem tax. In other words, the unit tax is equal to the ad-valorem tax multiplied by Ps. Therefore, the results obtained using the ad-valorem tax are equivalent to the ones obtained using the unit tax.

Exercise 3

3. **(Ramsey rule)** Consider a three-good economy ($k = 1, 2, 3$) in which every consumer has preferences represented by the utility function $U = x_1 + g(x_2) + h(x_3)$, where the functions $g(\cdot)$ and $h(\cdot)$ are increasing and strictly concave. Suppose that each good is produced with constant returns to scale from good 1, using one unit of good 1 per unit of good $k \neq 1$. Let good 1 be the numeraire and normalize the price of good 1 to equal 1. Let t_k denote the (specific) commodity tax on good k so the consumer price is $q_k = (1 + t_k)$.

   (a) Consider two commodity tax schemes $t = (t_1, t_2, t_3)$ and $t = (t'_1, t'_2, t'_3)$. Show that if $(1 + t'_k) = \phi(1 + t_k)$ for $k = 1, 2, 3$ for some scalar $\phi > 0$, then the two tax schemes raise the same amount of tax revenue.

   (b) Argue from part (a) that the government can without cost restrict tax schemes to leave one good untaxed.

   (c) Set $t_1 = 0$, and suppose that the government must raise revenue of R. What are the tax rates on goods 2 and 3 that minimize the welfare loss from taxation?

   (d) Show that the optimal tax rates are inversely proportional to the elasticity of the demand for each good. Discuss this tax rule.

   (e) When should both goods be taxed equally? Which good should be taxed more?

   Solution:

(a) The budget constraint for the consumer with tax scheme $t = (t_1, t_2, t_3)$ is:

$$(1 + t_2)x_2 + (1 + t_3)x_3 = -(1 + t_1)x_1$$

where the consumer sells units of the numeraire in order to purchase units of goods 2 and 3. (Note that all prices take into account taxes.) The above budget constraint can be alternatively writen as

$$(1 + t_1)x_1 + (1 + t_2)x_2 + (1 + t_3)x_3 = 0$$

Hence tax revenue R is:

$$R \equiv t_1x_1 + t_2x_2 + t_3x_3 = -(x_1 + x_2 + x_3)$$

Similar reasoning shows that with tax scheme $t' = (t'_1,\, t'_2,\, t'_3)$ the tax revenue R' is:

$$R' \equiv t'_1x'_1 + t'_2x'_2 + t'_3x'_3 = -(x'_1 + x'_2 + x'_3)$$

But the demand for each commodity is homogeneous of degree zero (Recall that, by proposition 3.D.2 in MWG, Walrasian demand is homogeneous of degree zero in prices). Hence, we have that

$$x_i = x_i(1 + t_1,\, 1 + t_2,\, 1 + t_3)$$

$$x_i = x_i(\phi(1 + t_1),\, \phi(1 + t_2),\, \phi(1 + t_3))$$
$$x_i = x_i(1 + t'_1,\, 1 + t'_2,\, 1 + t'_3) = x'_i$$

Therefore

$$(x_1 + x_2 + x_3) = (x'_1 + x'_2 + x'_3)$$

and $R = R'$.

(b) The value for ϕ can be chosen arbitrarily. In particular, a tax system with a tax t_k on good k can be shown to be equivalent to one with no tax on good k by choosing

$$\phi = \tfrac{1}{1+t_k}$$

(c) The optimization decision for the consumer is

$$\underset{\{x_1,x_2,x_3\}}{max}\; U = x_1 + g(x_2) + h(x_3)$$

$$\text{s.t. } x_1 + (1 + t_2)x_2 + (1 + t_3)x_3 = 0$$

Substituting the constraint into the objective function for x_1 reduce the F.O.C. to

$$g'(x_2) - (1 + t_2) = 0 \quad \text{and} \quad h'(x_3) - (1 + t_3) = 0$$

These necessary conditions result in demand functions $x_2 = x_2(1+t_2)$ and $x_3 = x_3(1+t_3)$, where x_2 is a function of the effective price of good 2, $1+t_2$, and similarly for the demand of good 3. Thus, the above budget constraint can be written as a function of these demands

$$x_1 = -(1+t_2)x_2 - (1+t_3)x_3.$$

The optimization of the government can now be written as

$$\max_{\{t_2,t_3\}} \; U = \underbrace{-(1+t_2)x_2 - (1+t_3)x_3}_{x_1} + g(x_2) + h(x_3)$$

$$\text{s.t. } R = t_2x_2 + t_3x_3$$

where x_2 and x_3 are, in turn, functions of $(1+t_2)$ and $(1+t_3)$ respectively.

The solution to this problem provides the tax rates that minimize welfare loss. The necessary conditions are:

$$g'x_2' - x_2 - (1+t_2)x_2' - \lambda(x_2 + t_2x_2') = 0$$

$$h'x_3' - x_3 - (1+t_3)x_3' - \lambda(x_3 + t_3x_3') = 0.$$

From the consumer's choice problem $g' = 1+t_2$ and $h' = 1+t_3$. These allow the implicit solutions

$$t_2 = -\tfrac{x_2}{x_2'}\tfrac{1+\lambda}{\lambda} \text{ and } t_3 = -\tfrac{x_3}{x_3'}\tfrac{1+\lambda}{\lambda}$$

(d) The elasticity of demand for good k is defined as

$$\varepsilon_k^d = \tfrac{(1+t_k)x_k'}{x_k}$$

by this definition,

$$\tfrac{t_2}{1+t_2} = -\tfrac{1}{\varepsilon_2^d}\tfrac{1+\lambda}{\lambda}$$

and

$$\tfrac{t_3}{1+t_3} = -\tfrac{1}{\varepsilon_3^d}\tfrac{1+\lambda}{\lambda}.$$

The tax rate on good k is therefore inversely proportional to the elasticity of demand for that good. Setting the relative taxes in this way minimizes the excess burden resulting from the need to raise the revenue R. This relationship between tax rates and elasticity is often referred to as the "inverse elasticity rule"

(e) This tax rule implies that the good with the lower elasticity of demand should have higher tax rate. The two goods should be taxed at the same rate only if they have the same elasticity of demand.

Exercise 4

4. Consider the utility function $U = \alpha\log(x_1) + \beta\log(x_2) - l$ and budget constraint $wl = q_1x_1 + q_2x_2$.

   (a) Show that the price elasticity of demand for both commodities is equal to -1.

   (b) Setting producer prices at $p_1 = p_2 = 1$, show that the inverse elasticity rule implies $\frac{t_1}{t_2} = \frac{q_1}{q_2}$.

   (c) Letting $w = 100$ and $\alpha + \beta = 1$, calculate the tax rates required to achieve revenue of $R = 10$.

Solution:

(a) The consumer's demands is solve

$$\max_{\{x_1,x_2,l\}} \alpha\log(x_1) + \beta\log(x_2) - l$$
$$\text{s.t. } wl = q_1x_1 + q_2x_2$$

or equivalently

$$\max_{\{x_1,x_2,l\}} \alpha\log(x_1) + \beta\log(x_2) - \left(\tfrac{q_1}{w}x_1 + \tfrac{q_2}{w}x_2\right)$$

The F.O.C.s are then

$$\tfrac{\alpha}{x_1} = \tfrac{q_1}{w} \text{ and } \tfrac{\beta}{x_2} = \tfrac{q_2}{w}$$

Then, the demands are

$$x_1 = \tfrac{\alpha w}{q_1} \text{ and } x_2 = \tfrac{\beta w}{q_2}$$

The elasticity of demand is defined by

$$\varepsilon_i^d = \tfrac{dx_i}{dq_i}\tfrac{q_i}{x_i}$$

Calculating this for good 1 obtains

$$\varepsilon_1^d = -\tfrac{\alpha w}{q_1^2}\tfrac{q_1}{(\frac{\alpha w}{q_1})} = -1$$

Calculating this for good 2 obtains

$$\varepsilon_2^d = -\tfrac{\beta w}{q_2^2}\tfrac{q_2}{(\frac{\beta w}{q_2})} = -1$$

(b) The inverse elasticity rule (see exercise 3) states that

$$\frac{t_i}{1+t_i} = \frac{\alpha-\lambda}{\lambda}\frac{1}{\varepsilon_i^d}, \; i = 1, 2.$$

Hence

$$\frac{t_1}{1+t_1}\varepsilon_1^d = \frac{\alpha-\lambda}{\lambda} = \frac{t_2}{1+t_2}\varepsilon_2^d$$

But $\varepsilon_i^d = -1$ and $1 + t_i = q_i$, so

$$\frac{t_1}{q_1} = \frac{t_2}{q_2} \text{ and rearranging we have } \frac{t_1}{t_2} = \frac{q_1}{q_2}.$$

(c) Revenue is defined by

$$R = t_1x_1 + t_2x_2$$

Using the solutions for the demands $x_1 = \frac{\alpha w}{q_1}$ and $x_2 = \frac{\beta w}{q_2}$ we have

$$R = t_1(\frac{\alpha w}{q_1}) + t_2(\frac{\beta w}{q_2})$$

Using the relation $\frac{t_1}{t_2} = \frac{q_1}{q_2}$ we just found in part b as $t_1 = \frac{q_1}{q_2}t_2$

$$R = (\frac{q_1}{q_2}t_2)(\frac{\alpha w}{q_1}) + t_2(\frac{\beta w}{q_2}) = w[(\frac{q_1}{q_2})(\frac{\alpha}{q_1}) + \frac{\beta}{q_2}]t_2 = \frac{w}{q_2}(\alpha + \beta)t_2$$

Finally, since $1 + t_i = q_i$,$\alpha + \beta = 1$, $R = 10$ and $w = 100$, the optimal tax on good 2 solves

$$10 = \frac{100}{1+t_2}t_2$$

which has solution $t_2 = \frac{1}{9}$, and hence $t_1 = \frac{1}{9}$.

Exercise 5

5. Let the consumer have the utility function $U = x_1^{\rho_1} + x_2^{\rho_2} - l$.

   (a) Show that the utility maximizing demands are $x_1 = \left[\frac{\rho_1 w}{q_1}\right]^{1/[1-\rho_1]}$ and $x_2 = \left[\frac{\rho_2 w}{q_2}\right]^{1/[1-\rho_2]}$.

   (b) Letting $p_1 = p_2 = 1$, use the inverse elasticity rule to show that the optimal tax rates are related by $\frac{1}{t_2} = \left[\frac{\rho_2-\rho_1}{1-\rho_2}\right] + \left[\frac{1-\rho_1}{1-\rho_2}\right]\frac{1}{t_1}$.

   (c) Setting $w = 100$, $\rho_1 = 0.75$, $\rho_2 = 0.5$, find the tax rates required to achieve revenue of $R = 10$ and $R = 300$.

   (d) Calculate the proportional reduction in demand for the two goods comparing the no-tax position with the position after introduction of the optimal taxes for both revenue levels. Comment on the results.

Solution:

(a) If the consumer maximization problem is $max\ U = x_1^{\rho_1} + x_2^{\rho_2} - l$ s.t. $q_1x_1 + q_2x_2 = wl$

Thus we can rewrite the budget constraint $l = \frac{q_1x_1+q_2x_2}{w}$ and we can replace into the utility function for an unconstrained optimization problem as:

$$\max\ U = x_1^{\rho_1} + x_2^{\rho_2} - \frac{q_1x_1}{w} - \frac{q_2x_2}{w}$$

Taking first order condtions with respect to every good, x_i, yields

$$\frac{\partial U}{\partial x_i} = \rho_i x_i^{\rho_i-1} - \frac{q_i}{w} = 0$$

and rearranging, we obtain

$$\rho_i x_i^{\rho_i-1} = \frac{q_i}{w}$$

Solving for x_i we get the utility maximizing demands as required.

$$x_i = \left(\frac{w\rho_i}{q_i}\right)^{\frac{1}{1-\rho_i}}$$

(b) The first step is to calculate the price elasticity using the demand function we just found:

$$\varepsilon_i^d = -\frac{1}{1-\rho_i}$$

With $p_1 = p_2 = 1$ the **inverse elasticity rule** states that (see previous exercise):

$$\frac{t_1}{1+t_1}\varepsilon_1^d = \frac{t_2}{1+t_2}\varepsilon_2^d \text{ or } \frac{1+t_2}{t_2}\varepsilon_1^d = \frac{1+t_1}{t_1}\varepsilon_2^d$$

Substituting for the elasticities

$$\frac{1+t_2}{t_2}\left(-\frac{1}{1-\rho_1}\right) = \frac{1+t_1}{t_1}\left(-\frac{1}{1-\rho_2}\right)$$

$$\begin{aligned}
\frac{1}{t_2}\left(\frac{1+t_2}{1-\rho_1}\right) &= \frac{1}{t_1}\left(\frac{1+t_1}{1-\rho_2}\right) \\
\frac{1-\rho_2}{t_2}(1+t_2) &= \frac{1-\rho_1}{t_1}(1+t_1) \\
\frac{1-\rho_2}{t_2} + \frac{1-\rho_2}{t_2}t_2 &= \frac{1-\rho_1}{t_1} + \frac{1-\rho_1}{t_1}t_1 \\
\frac{1-\rho_2}{t_2} &= \frac{1-\rho_1}{t_1} + \frac{1-\rho_1}{t_1}t_1 - \frac{1-\rho_2}{t_2}t_2 \\
\frac{1-\rho_2}{t_2} &= \frac{1-\rho_1}{t_1} + (1-\rho_1) - (1-\rho_2)
\end{aligned}$$

finally

$$\frac{1}{t_2} = \frac{1-\rho_1}{1-\rho_2}\cdot\frac{1}{t_1} + \frac{\rho_2-\rho_1}{1-\rho_2} \text{ or } \frac{1}{t_2} = \left[\frac{\rho_2-\rho_1}{1-\rho_2}\right] + \left[\frac{1-\rho_1}{1-\rho_2}\right]\frac{1}{t_1}.$$

(c) Using the parameter values gives

$$\frac{1}{t_2} = -0.5 + 0.5\frac{1}{t_1}$$

so

$$t_2 = \frac{2}{\frac{1}{t_1}-1}$$

Then, given the revenue constraint

$$R = t_1x_1 + t_2x_2$$

but we know the optimal values for the demand and using the fact that $1+t_i = q_i$, then

$$\begin{aligned}
R &= t_1\left(\frac{w\rho_1}{q_1}\right)^{1/(1-\rho_1)} + t_2\left(\frac{w\rho_2}{q_2}\right)^{1/(1-\rho_2)} \\
R &= t_1\left(\frac{w\rho_1}{1+t_1}\right)^{1/(1-\rho_1)} + t_2\left(\frac{w\rho_2}{1+t_2}\right)^{1/(1-\rho_2)}
\end{aligned}$$

and t_2 is also known, then

$$R = t_1 \left(\frac{w\rho_1}{1+t_1}\right)^{1/(1-\rho_1)} + \left(\frac{2}{\frac{1}{t_1}-1}\right)\left(\frac{w\rho_2}{1+\left(\frac{2}{\frac{1}{t_1}-1}\right)}\right)^{1/(1-\rho_2)}$$

Replacing the values of the parameters

$$R = t_1 \left(\frac{75}{1+t_1}\right)^4 + \left(\frac{2}{\frac{1}{t_1}-1}\right)\left(\frac{25}{1+\left(\frac{2}{\frac{1}{t_1}-1}\right)}\right)^2$$

which simplifies to

$$R = 625\left[\frac{50625t_1}{(1+t_1)^4} + \frac{16t_1^4}{(x^2-1)^2}\right] \tag{1}$$

The revenue curve has a maximum level of revenue around $t_1 = 0.4$, this is known as Laffer property. For $R = 10$ the solution is $t_1 = 0.0031$ and $t_2 = 0.0062$; as depicted in the following figure which illustrates expression (1) evaluated at $R = 10$ as a function of t_1. In the case $R = 300$ the solution is $t_1 = 0.1814$ and $t_2 = 0.4431$.

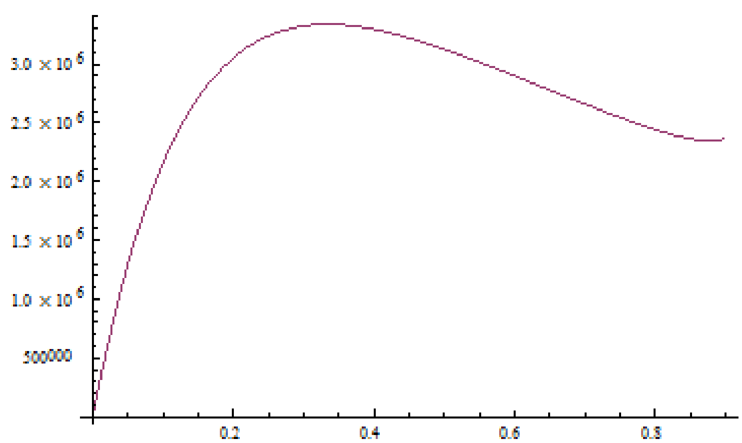

Figure 2. Tax revenue $R = 10$ as a function of t_1.

(d) The proportional reduction in demand for the two goods comparing the no-tax position with the position after introduction of the optimal taxes for both revenue levels is in the next table.

R	x_1	%	x_2	%
0	3,164	–	25	–
10	3,125	1.23	24.69	1.24
300	1,624	48.67	12.00	52.00

As we can see, the optimal taxes do reduce demand in approximately the same proportion for both commodities. In this case the interpretation of the Ramsey rule is applicable even when the tax intervention has a significant effect on the level of demand.