# EconS 501 - Micro Theory I

# Recitation #6 - Production Theory-II[1]

## Exercise 1

**Exercise 5.C.11** Show that $\frac{\partial z_\ell(w,q)}{\partial q} > 0$ if and only if marginal cost at $q$ is increasing in $w_\ell$.

- Assume that $c(\cdot)$ is twice continuously differentiable. By Proposition 5.C.2(vi), $z(\cdot)$ is continuously differentiable and

$$\frac{\partial z_\ell (w,q)}{\partial q} = \left(\frac{\partial}{\partial q}\right)\left(\frac{\partial c(w,q)}{\partial w_\ell}\right) = \left(\frac{\partial}{\partial w_\ell}\right)\left(\frac{\partial c(w,q)}{\partial q}\right).$$

1. - Hence

$$\frac{\partial z_\ell (w,q)}{\partial q} > 0$$

if and only if

$$\left(\frac{\partial}{\partial w_\ell}\right)\left(\frac{\partial C(w,q)}{\partial q}\right) > 0,$$

that is, marginal cost is increasing in $w_\ell$.

## Exercise 2

**Exercise 5.C.13** A price-taking firm produces output $q$ from inputs $z_1$ and $z_2$ according to a differentiable concave production function $f(z_1, z_2)$. The price of its output is $p > 0$, and the prices of its inputs are $(w_1, w_2) >> 0$. However, there are two unusual things about this firm. First, rather than maximizing profit, the firm maximizes revenue (the manager wants her firm to have bigger dollar sales than any other). Second, the firm is cash constrained. In particular, it has only $C$ dollars on hand before production and, as a result, its total expenditures on inputs cannot exceed $C$.

Suppose one of your econometrician friends tells you that she has used repeated observations of the firm's revenues under various output prices, input prices, and levels of the financial constraint and has determined that the firm's revenue level $R$ can be expressed as the following function of the variables $(p, w_1, w_2, C)$:

$$R(p, w_1, w_2, C) = p\left[\gamma + \ln C - \alpha \ln w_1 - (1-\alpha)\ln w_2\right].$$

($\gamma$ and $\alpha$ are scalars whose values she tells you.) What is the firm's use of input $z_1$ when prices are $(p, w_1, w_2)$ and it has $C$ dollars of cash on hand?

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- Denote the production function of the firm by $f(\cdot)$. Then its optimization problem is

$$\max_{(z_1,z_2)\geq 0} \quad p\cdot f(z_1,z_2).$$

$$\text{subject to } \ w_1 z_1 + w_2 z_2 \leq C$$

This is analogous to the utility maximization problem in Section 3.D and the function $R(\cdot)$ corresponds to the indirect utility function. Hence, analogously to Roy's identity (Proposition 3.G.4), the input demands are obtained as

$$-\frac{1}{\nabla_C R(p,w,C)}\nabla_w R(p,w,C) = \left(\frac{\alpha C}{w_1}, \frac{(1-\alpha)C}{w_2}\right) = (z_1,z_2).$$

# Exercise 3

**Exercise 5.D.4** Consider a firm that has a distinct set of inputs and outputs. The firm produces $M$ outputs; let $q=(q_1,...,q_M)$ denote a vector of its output levels. Holding factor prices fixed, $C(q_1,...,q_M)$ is the firm's cost function. We say that $C(\cdot)$ is *subaddittive* if for all $(q_1,...,q_M)$, there is no way to break up the production of amounts $(q_1,...,q_M)$ among several firms, each with cost function $C(\cdot)$, and lower the costs of production. That is, there is no set of, say $J$ firms and collection of production vectors $\{q_j=(q_{1j},...,q_{Mj})\}_{j=1}^J$ such that

$$\sum_j q_j = q \ \text{ and } \ \sum_j C(q_j) < C(q).$$

When $C(\cdot)$ is subadditive, it is usual to say that the industry is a *natural monopoly* because production is cheapest when it is done by only one firm.

a. Consider the single-output case, $M=1$. Show that if $C(\cdot)$ exhibits decreasing average costs, then $C(\cdot)$ is subadditive.

   - Suppose that $q=\sum_{j=1}^J q_j$. By the decreasing average costs (and $C(0)=0$), $\left(\frac{q_j}{q}\right)C(q)\leq C(q_j)$. By summing over $j$, we obtain $C(q)\leq\sum_{j=1}^J C(q_j)$. Hence there is no way to break up the production of $q$ among multiple firms and lower the cost of production. Hence $C(\cdot)$ is subadditive.

b. Now consider the multiple-output case, $M>1$. Show by example that the following multiple-output extension of the decreasing average cost assumption is *not* sufficient for $C(\cdot)$ to be subadditive:

$$C(\cdot) \text{ exhibits } \textit{decreasing ray average} \text{ cost if for any } q\in\mathbb{R}_+^M,$$

$$C(q) > \frac{C(kq)}{k} \text{ for all } k>1.$$

- Let $M = 2$ and define cost function $C(q) = \sqrt{\min\{q_1, q_2\}}$. Then $C(\cdot)$ exhibits decreasing ray from the origin, i.e., decreasing average cost, as required. However, let $q_1 = (1, 8)$, $q_2 = (8, 1)$, and $q = q_1 + q_2 = (9, 9)$. Therefore, $C(q_1) = C(q_2) = 1$ and $C(q) = 3$, which implies

$$C(q) > C(q_1) + C(q_2)$$

Hence, cost function $C(q) = \sqrt{\min\{q_1, q_2\}}$, despite exhibiting decreasing average cost, is *not* subadditive.

# Exercise 4

**Exercise 5.E.5 (MWG) [Output distribution between two plants.]** Suppose that a firm owns two plants, each producing the same good. Every plant $j$'s average cost is given by

$$AC_j(q_j) = \alpha + \beta_j q_j \text{ for } q_j \geq 0, \text{ where } j = \{1, 2\}$$

where coefficient $\beta_j$ may differ from plant to plant, i.e., if $\beta_1 > \beta_2$ plant 2 is more efficient than plant 1 since its average costs increase less rapidly in output. Assume that you are asked to determine the cost-minimizing distribution of aggregate output $q = q_1 + q_2$, among the two plants (i.e., for a given aggregate output $q$, how much $q_1$ to produce in plant 1 and how much $q_2$ to produce in plant 2.) For simplicity, consider that aggregate output $q$ satisfies $q < \frac{\alpha}{\max_j |\beta_j|}$. (You will be using this condition in part b.)

a) If $\beta_j > 0$ for every plant $j$, how should output be located among the two plants?

- The cost-minimization problem in which we find the optimal combination of output $q_1$ and $q_2$ that minimizes the total cost of production across plants is

$$\min_{q_1, q_2} \; TC_1(q_1) + TC_2(q_2)$$

$$\text{subject to } \; q_1 + q_2 = q$$

or equivalently, the profit maximization problem in which firms choose the optimal combination of output $q_1$ and $q_2$ that maximizes the total profits across all plants is

$$\max_{q_1, q_2} \; \underbrace{pq_1 - TC_1(q_1)}_{\pi_1} + \underbrace{pq_2 - TC_2(q_2)}_{\pi_2}$$

$$\text{subject to } \; q_1 + q_2 = q$$

- If the average cost is $AC_j(q_j) = \alpha + \beta_j q_j$ then the total cost is $TC_j(q_j) = (\alpha + \beta_j q_j)q_j$. Thus, we can rewrite the above PMP as:

$$\max_{q_1, q_2} \; pq_1 - (\alpha + \beta_1 q_1)q_1 + pq_2 - (\alpha + \beta_2 q_2)q_2$$

$$\text{subject to } \; q_1 + q_2 = q$$

Taking first order conditions with respect to $q_1$ and $q_2$ yields

$$\frac{\partial(\pi_1 + \pi_2)}{\partial q_1} = p - \alpha - 2\beta_1 q_1 = \lambda$$

$$\frac{\partial(\pi_1 + \pi_2)}{\partial q_2} = p - \alpha - 2\beta_2 q_2 = \lambda$$

$$\frac{\partial(\pi_1 + \pi_2)}{\partial \lambda} = q_1 + q_2 = q$$

Using the first two order conditions, we obtain

$$p - \alpha - 2\beta_1 q_1 = p - \alpha - 2\beta_2 q_2$$

and rearranging, $q_2 = \frac{\beta_2}{\beta_1} q_1$. Replacing this expression into the constraint $q_1 + q_2 = q$ yields

$$q_1 + \underbrace{\frac{\beta_1}{\beta_2} q_1}_{q_2} = q$$

and solving for $q_1$ entails the cost-minimizing production in plant 1,

$$q_1 \left(1 + \frac{\beta_1}{\beta_2}\right) = q, \quad \text{thus} \quad q_1 = \frac{\beta_2}{\beta_1 + \beta_2} q,$$

and operating similarly for $q_2$, we find

$$q_2 = \frac{\beta_1}{\beta_1 + \beta_2} q$$

- *Extension:* Note that, generally for $J$ plants, the average cost of plant $j$ is $AC_j(q_j) = \alpha + \beta_j q_j$ implying that the total cost must be $TC_j(q_j) = (\alpha + \beta_j q_j) q_j$. Therefore, plant $j$'s marginal cost is $MC_j(q_j) = \alpha + 2\beta_j q_j$. Since $\beta_j > 0$ for every $j$, the first order necessary and sufficient conditions for cost minimization are: (1) that firms' marginal costs coincide (otherwise, we would still have incentives to distribute a larger production to those firms with the lowest marginal cost)

$$MC_j(q_j) = MC_{j'}(q_{j'}) \text{ for any two plants } j \text{ and } j'$$

and; (2) that the aggregate output constraint holds

$$q_1 + q_2 + ...q_J = q.$$

From these conditions we obtain

$$q_j = \frac{\frac{q}{\beta_j}}{\sum_h \frac{1}{\beta_h}}.$$

which coincides with our results for $N = 2$ plants,

$$q_1 = \frac{\frac{q}{\beta_1}}{\frac{1}{\beta_1} + \frac{1}{\beta_2}} = \frac{\beta_2}{\beta_1 + \beta_2} q.$$

The next figure depicts the average and marginal cost curves for two plants satisfying $\beta_2 > \beta_1$. In particular, the firm manager chooses, for a given aggregate output $q = q_1 + q_2$, the individual output levels $q_1$ and $q_2$ that equate the marginal costs across both plants (see vertical axis).

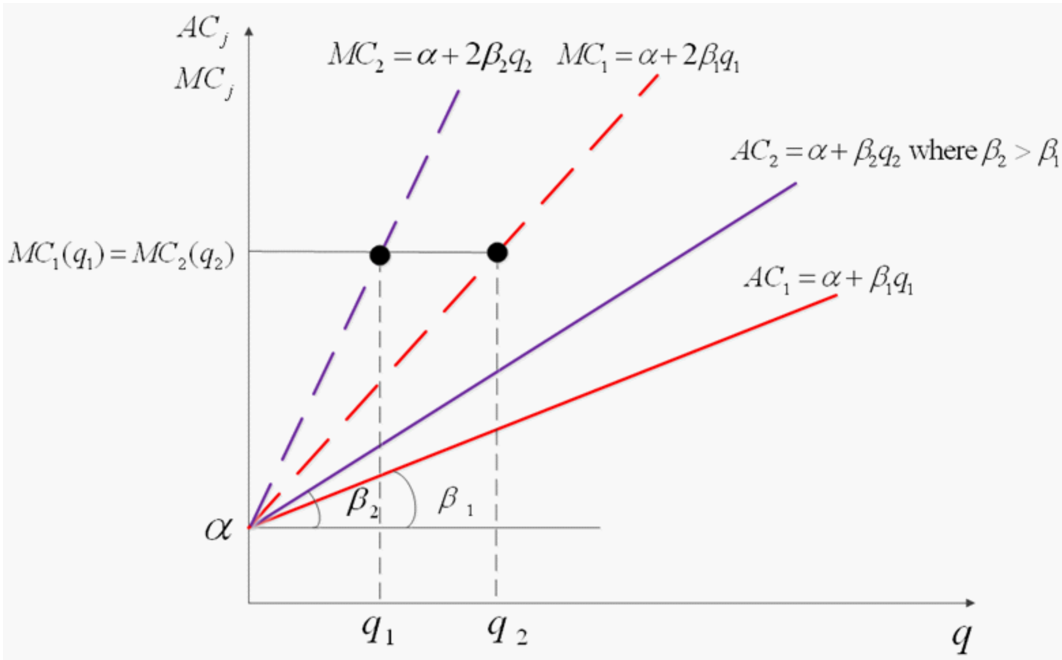


Figure 1. $\beta_j > 0$ for every firm.

b) If $\beta_j < 0$ for every plant $j$, how should output be located among the two plants?

- First, note that $\beta_j < 0$ implies that the average cost $AC_j(q_j) = \alpha + \beta_j q_j$ is decreasing in output. Hence, it is cost-minimizing to concentrate all production on the plant with the smallest $\beta_j < 0$ (the most negative $\beta_j$) because average costs (and total costs) are the minimized by doing so.
- The next figure depicts a firm in which both plants exhibit decreasing average costs, but $\beta_2 < \beta_1 < 0$, implying that it is beneficial for the firm to concentrate all output in plant 2. In addition, note that the average cost in plant 1 is positive for all $q_1$ as long as $\alpha - \beta_1 q_1 > 0$, or $q_1 < \frac{\alpha}{\beta_1}$, where $\frac{\alpha}{\beta_1}$ represents the horizontal intercept of $AC_1$ in the figure. Similarly for firm 2, where $AC_2 > 0$ for all $q_2$ as long as $q_2 < \frac{\alpha}{\beta_2}$, where $\frac{\alpha}{\beta_2}$ represents the horizontal intercept of $AC_2$. Hence, the original condition $q < \frac{\alpha}{\max_j |\beta_j|}$ is equivalent to $q < \min_j \frac{\alpha}{|\beta_j|}$, graphically implying that the aggregate output $q$ lies to the left-hand side to the smallest horizontal

intercept.

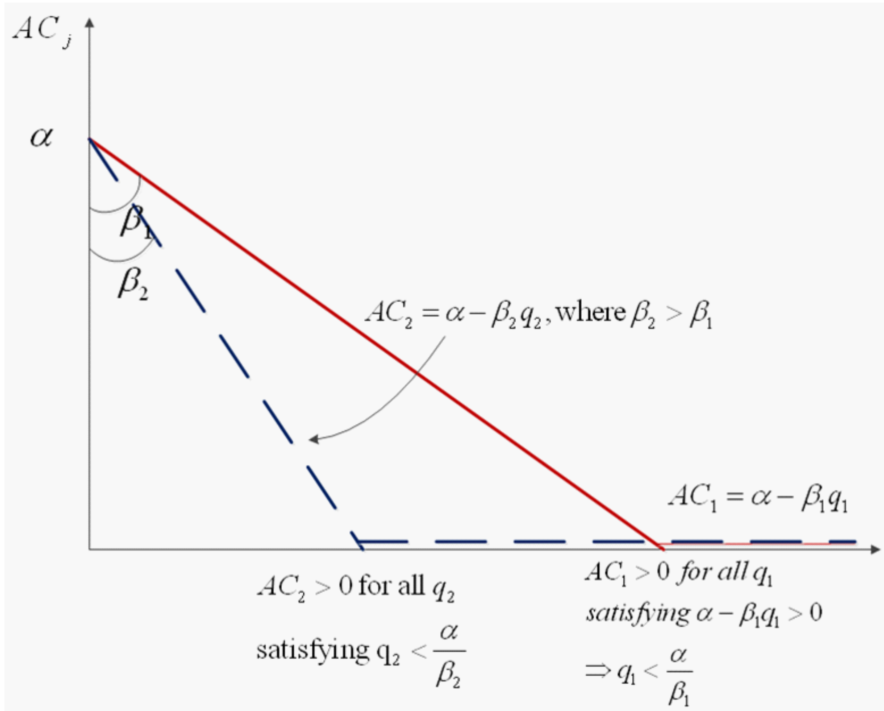


Figure 2. $\beta_j < 0$ for every firm.

c) If $\beta_j > 0$ for some plants and $\beta_j < 0$ for others?

- Similarly as in part (b), the firm now faces some plants with increasing average costs (those with $\beta_j > 0$) and some plants with decreasing average costs (those with $\beta_j < 0$). Hence, it is cost-minimizing to concentrate all production on the plant/s with the smallest $\beta_j < 0$, since it benefits from the most rapidly decreasing average costs. The next figure depicts a firm with plant 1 (2) having increasing (decreasing, respectively) average costs.

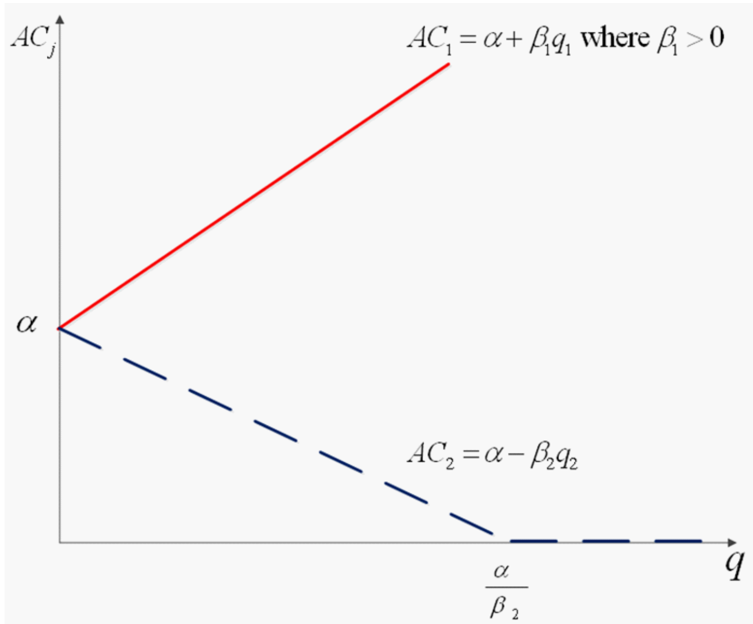


Figure 3. $\beta_1 > 0$ but $\beta_2 < 0$

# Exercise 5

Assume there is a firm that has a regular production function given by $q = f(z_1, ..., z_n)$ with Constant Returns to Scale (CRS). Show that if the firm pays the use of each input according to its exact marginal productivity, then profits are equal to zero.

**Solution:** Since the production function shows CRS we know that it is homogeneous of degree one, then, according to Euler's theorem for homogeneous functions:

$$\frac{\partial f(\mathbf{z})}{\partial z_1} z_1 + \frac{\partial f(\mathbf{z})}{\partial z_2} z_2 + ... + \frac{\partial f(\mathbf{z})}{\partial z_n} z_n = f(z_1, ..., z_n) \tag{1}$$

Then, if the firm pays each input according to the market value of the marginal productivity of each input:

$$p\tfrac{\partial f(\mathbf{z})}{\partial z_i} = w_i :\Rightarrow: \tfrac{\partial f(\mathbf{z})}{\partial z_i} = \tfrac{w_i}{p}, \text{ for every input } i$$

Using this result on (1),

$$\tfrac{w_1}{p} z_1 + \tfrac{w_2}{p} z_2 + ... + \tfrac{w_n}{p} z_n = f(z_1, ..., z_n)$$

$$w_1 z_1 + w_2 z_2 + ... + w_n z_n = pf(z_1, ..., z_n)$$

$$pf(z_1, ..., z_n) - (w_1 z_1 + w_2 z_2 + ... + w_n z_n) = 0$$

Since $pf(z_1, ..., z_n)$ is the total revenue of the firm and $(w_1 z_1 + w_2 z_2 + ... + w_n z_n)$ is the total cost, then this difference is the total profit of the firm, which is zero, as we wanted to show.