# EconS 501 - Microeconomic Theory I
# Recitation #5 - Production Theory-I[1]

## Exercise 1

1. **Exercise 5.B.2 (MWG):** Suppose that $f(\cdot)$ is the production function associated with a single-output technology, and let $Y$ be the production set of this technology. Show that $Y$ satisfies constant returns to scale if and only if $f(\cdot)$ is homogeneous of degree one.

   - Suppose first that a production set $Y$ exhibits constant returns to scale (see figure below). Let $z \in \mathbb{R}_+^{L-1}$ and $\alpha > 0$. Then $(-z,\ f(z)) \in Y$. By constant returns to scale, $(-\alpha z, \alpha f(z)) \in Y$. Hence $\alpha f(z) \leq f(\alpha z)$.

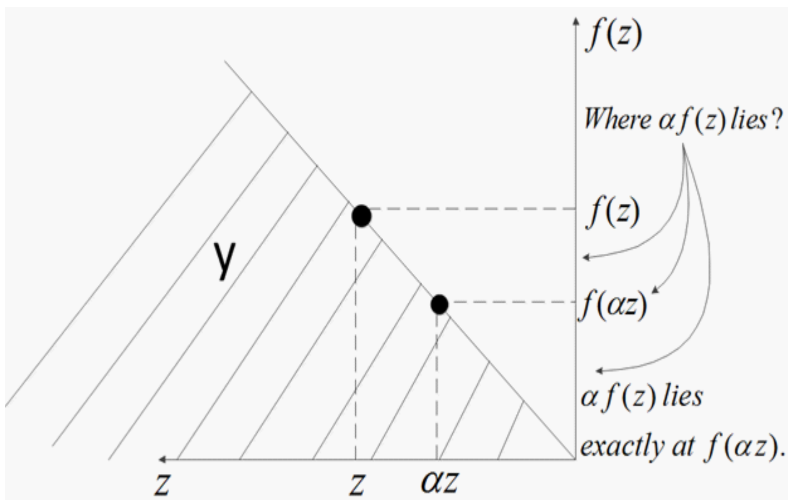


Figure 1. Constant returns to scale.

   - By applying this inequality to $\alpha z$ in place of $z$ and $\frac{1}{\alpha}$ in place of $\alpha$, we obtain

$$\frac{1}{\alpha} f(\alpha z) \leq f\left(\frac{1}{\alpha}(\alpha z)\right) = f(z), \text{ or } f(\alpha z) \leq \alpha f(z)$$

     Hence $f(\alpha z) = \alpha f(z)$. The homogeneity of degree one is thus obtained.

   - Suppose conversely that $f(\cdot)$ is homogeneous of degree one. Let $(-z, q) \in Y$ and $\alpha \geq 0$, then $q \leq f(z)$ and hence $\alpha q \leq \alpha f(z) = f(\alpha z)$. Since $(-\alpha z, f(\alpha z)) \in Y$, we obtain $(-\alpha z, \alpha q) \in Y$. The constant returns to scale is thus obtained.

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# Exercise 2

2. **Exercise 5.B.3 (MWG):** Show that for a single-output technology, the production set $Y$ is a convex if and only if the production function $f(z)$ is concave.

- In order to prove this "if and only if statement" we need to show first that: if the production set $Y$ is convex, then the production function $f(z)$ is concave. And second, we need to show the converse: that if the production function $f(z)$ is concave then the set $Y$ must be convex.

- First, suppose that $Y$ is convex. Let $z, z' \in \mathbb{R}_+^{L-1}$ and $\alpha \in [0,1]$, then $(-z, f(z)) \in Y$ and $(-z', f(z')) \in Y$. By the convexity,

$$(-(\alpha z + (1-\alpha) z),\ \alpha f(z) + (1-\alpha) f(z)) \in Y.$$

  Thus, $\alpha f(z) + (1-\alpha) f(z) \leq f(\alpha z + (1-\alpha) z)$. Hence $f(z)$ is concave.

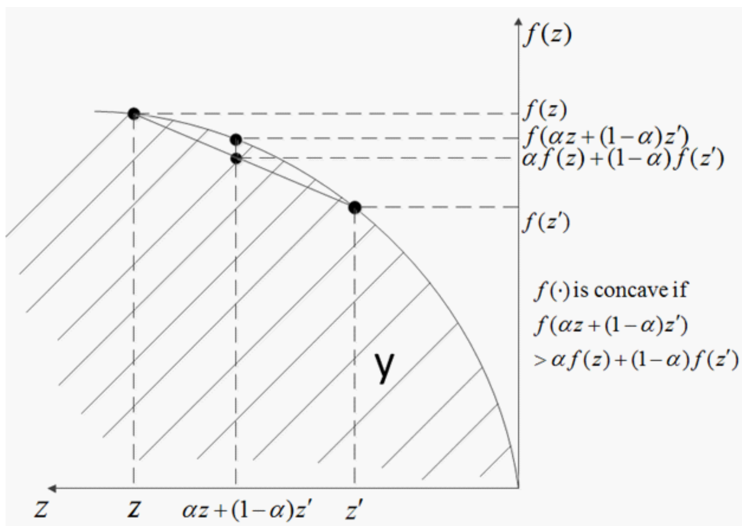



Figure 2. Decreasing returns to scale.

- Let us now suppose that $f(z)$ is concave. Let $(q, -z) \in Y$, $(q', -z') \in Y$, and $\alpha \in [0,1]$, then $q \leq f(z)$ and $q' \leq f(z')$. Hence

$$\alpha q + (1-\alpha) q' \leq \alpha f(z) + (1-\alpha) f(z')$$

  By concavity,

$$\alpha f(z) + (1-\alpha) f(z') \leq f(\alpha z + (1-\alpha) z')$$

  Thus

$$\alpha q + (1-\alpha) q' \leq f(\alpha z + (1-\alpha) z')$$

  Hence

$$(-(\alpha z + (1-\alpha) z'), \alpha q + (1-\alpha) q') = \alpha(-z, q) + (1-\alpha)(-z', q') \in Y$$

  Therefore $Y$ is convex.

- Finally, note that if, in contrast, the production function $f(z)$ is convex in $z$ (i.e., the production process exhibits increasing returns to scale), the production set $Y$ would be that in figure 3.

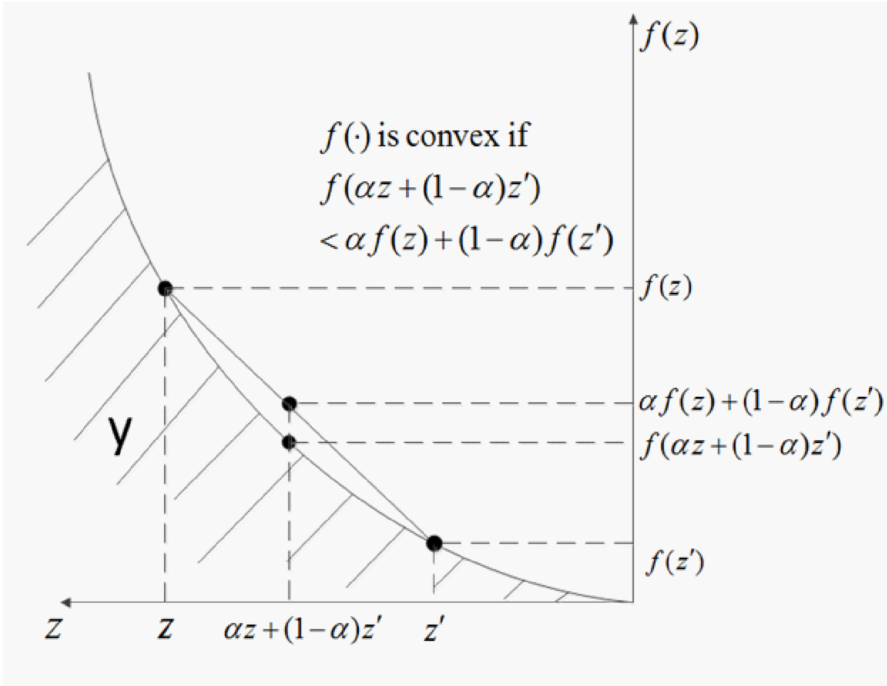



Figure 3. Increasing returns to scale.

# Exercise 3

3. Given a CES (Constant Elasticity of Substitution) production function:

$$q = f(z_1,\, z_2) = A\left[\delta z_1^{\rho} + (1-\delta) z_2^{\rho}\right]^{\frac{1}{\rho}}\ , \text{ where } A > 0 \text{ and } 0 < \delta < 1$$

Calculate the Marginal Rate of Technical Substitution (MRTS) and the Elasticity of Substitution ($\sigma$), where $\sigma \equiv \frac{d\ln\left(\frac{z_2}{z_1}\right)}{d\ln MRTS}$.

(a) Is it an homogeneous production function?

(b) Show, using MRTS and $\sigma$, that:

1. when $\rho \to -\infty$ the CES production function represents the Leontief production function;
2. when $\rho = 1$ the CES production function represents a perfect substitutes technology; and
3. when $\rho = 0$ the CES production function represent a Cobb-Douglas technology.

**Solution:**

a) We can calculate the MRTS between the two factors as:

$$MRTS_1^2 = \frac{\frac{\partial q}{\partial z_1}}{\frac{\partial q}{\partial z_2}} = \frac{\left(\frac{1}{\rho}\right) A\left[\delta z_1^\rho + (1-\delta) z_2^\rho\right]^{\frac{1}{\rho}-1} \rho\delta z_1^{\rho-1}}{\left(\frac{1}{\rho}\right) A\left[\delta z_1^\rho + (1-\delta) z_2^\rho\right]^{\frac{1}{\rho}-1} \rho(1-\delta) z_2^{\rho-1}} = \frac{\delta z_1^{\rho-1}}{(1-\delta) z_2^{\rho-1}}$$

Using the definition of the Elasticity of Substitution we have:

$$\sigma = \frac{d\ln\left(z_2/z_1\right)}{d\ln MRTS} = \frac{d\ln\left(z_2/z_1\right)}{d\ln\left(\frac{\partial q/\partial z_1}{\partial q/\partial z_2}\right)}$$

to find this expression we can use the expression of the MRTS we just found:

$$MRTS = \frac{\delta}{(1-\delta)}\left(\frac{z_2}{z_1}\right)^{1-\rho}$$

and using a logarithmic transformation

$$\ln(MRTS) = \ln\left(\frac{\delta}{1-\delta}\right) + (1-\rho)\ln\left(\frac{z_2}{z_1}\right)$$

solving for $\ln(z_2/z_1)$ we have:

$$\ln\left(\frac{z_2}{z_1}\right) = \left(\frac{1}{1-\rho}\right)\left[\ln(MRTS) - \ln\left(\frac{\delta}{1-\delta}\right)\right]$$

thus,

$$\sigma = \frac{d\ln\left(\frac{z_2}{z_1}\right)}{d\ln MRTS} = \frac{1}{1-\rho}$$

As we can observe, the elasticity of substitution, $\sigma$, is a constant value for any production process and any output value. This is the reason for the name of the CES function.

b) To verify that it is an homogeneous production function, we increase all inputs by $\theta$:

$$f(\theta z_1, \theta z_2) = A\left[\delta\left(\theta z_1\right)^\rho + (1-\delta)\left(\theta z_2\right)^\rho\right]^{\frac{1}{\rho}} = A\left[\delta\theta^\rho\left(z_1\right)^\rho + (1-\delta)\theta^\rho\left(z_2\right)^\rho\right]^{\frac{1}{\rho}}$$

and rearranging

$$f(\theta z_1, \theta z_2) = \theta A\left[\delta\left(z_1\right)^\rho + (1-\delta)\left(z_2\right)^\rho\right]^{\frac{1}{\rho}} = \theta f(z_1, z_2)$$

then the function is homogeneous of degree one.

c) Now we analyze what happens for different values of parameter $\rho$ :

i) **When $\rho \to -\infty$:** From part $(a)$ we have $MRTS_1^2 = \frac{\delta z_1^{\rho-1}}{(1-\delta) z_2^{\rho-1}}$, the limit of MRTS when $\rho \to -\infty$ is

$$\lim_{\rho \to -\infty} MRTS_1^2 = \lim_{\rho \to -\infty} \frac{\delta z_1^{\rho-1}}{(1-\delta) z_2^{\rho-1}} = \frac{\delta}{(1-\delta)} \left(\frac{z_2}{z_1}\right)^{\infty}$$

As we can see, if $z_2 > z_1$ the MRTS goes to $\infty$ (indicating a vertical segment of the isoquant) if $z_2 < z_1$ the MRTS goes to zero (indicating a flat segment of the isoquant). These values of the MRTS are the same of the Leontief or Fixed Proportions production function.

ii) **When $\rho = 1$:** In this case the production function becomes

$$q = f(z_1, z_2) = A\delta z_1 + A(1-\delta) z_2$$

this production function is a perfect substitutes inputs technology (or $\sigma = \frac{\delta}{1-\delta}$, which is constant in $z_1$ and $z_2$).

iii) **When $\rho \to 0$:** The MRTS is

$$MRTS = \frac{\delta}{(1-\delta)} \left(\frac{z_2}{z_1}\right),$$

In the extreme case where $\rho = 0$, the elasticity of substitution $\sigma = \frac{1}{1-\rho}$ becomes $\sigma = 1$, thus coinciding with the elasticity of substitution of a Cobb-Douglas production function.

# Exercise 4

4. Assume a standard technology represented by the production function $q = f(z_1, z_2)$, show:

   (a) If the function represents always Constant Returns to Scale (CRS) , it is true that marginal productivity of the factors are constant along the same production process?

   (b) What if the degree of the production function is different from one?

**Solution:**

a) If the production function has CRS, then the function is homogeneous of degree one, thus, the marginal productivity of the factors (first derivatives of the production function) are also homogeneous, but one degree less than the original function (degree zero).

$$f_1(\theta z_1, \theta z_2) = \theta^0 f_1(z_1, z_2) = f_1(z_1, z_2)$$
$$f_2(\theta z_1, \theta z_2) = \theta^0 f_2(z_1, z_2) = f_2(z_1, z_2)$$

Since a ray from the origin increases both inputs by a common factor $\theta$, the input ratio $\frac{z_1}{z_2}$ remains constant in all points along the ray. In addition, the above result indicates that the marginal product of every input is constant along a given ray (i.e., for production plans using the same ratio of inputs $\frac{z_1}{z_2}$). Note that since the marginal product of every input is constant along a given ratio of input combinations $\frac{z_1}{z_2}$, then we must have that the ratio of marginal products $\frac{f_1(z_1, z_2)}{f_2(z_1, z_2)}$ is also constant along a given ray $\frac{z_1}{z_2}$. Finally, since

$$\frac{f_1(z_1,\, z_2)}{f_2(z_1,\, z_2)} = MRTS(z_1,\, z_2)$$

then the MRTS between inputs 1 and 2 is constant along a given ray $\frac{z_1}{z_2}$. Therefore, the production function is homothetic.

b) If the production function is homogeneous of degree $k \neq 1$, then by Euler's theorem we know that the marginal product of every input is homogeneous of degree $k - 1$. That is[2]

$$\begin{aligned} f_1(\theta z_1,\, \theta z_2) &= \theta^{k-1} f_1(z_1,\, z_2) \ \text{ for the marginal product of input 1, and} \\ f_2(\theta z_1,\, \theta z_2) &= \theta^{k-1} f_2(z_1,\, z_2) \ \text{ for the marginal product of input 2} \end{aligned}$$

In this case, the marginal product of every input is *not* constant along a given ray $\frac{z_1}{z_2}$ (in which we increase both $z_1$ and $z_2$ keeping their proportion $\frac{z_1}{z_2}$ unmodified). However, the production function is still homothetic since:

$$MRTS(\theta z_1,\, \theta z_2) = \frac{\theta^{k-1} f_1(z_1,\, z_2)}{\theta^{k-1} f_2(z_1,\, z_2)} = \frac{f_1(z_1,\, z_2)}{f_2(z_1,\, z_2)} = MRTS(z_1,\, z_2)$$

# Exercise 5

5. Obtain the conditional factor demand functions, $z(w, q)$, the cost function, $c(w, q)$, supply correspondence, $q(w, p)$, and profit function, $\pi(w, p)$, for the technology: $q = z_1^{\alpha} z_2^{\beta}$ with $\alpha, \beta \geq 0$.

**Solution:** The technology is a Cobb-Douglas production function, then, the conditional factor demand functions can be calculated as the solution of the costs minimization problem:

$$\min_{z_1, z_2} \ w_1 z_1 + w_2 z_2$$

$$\text{subject to} \quad z_1^{\alpha} z_2^{\beta} \geq q$$

[2] For instance, a firm with a production function that is homogeneous of degree $k = 2$, the firm experiences increasing returns to scale. In contrast, a production function homogeneous of degree $k = 1/3$ implies that the firm experiences decreasing returns to scale.

The next figure depicts the cost-minimization problem.

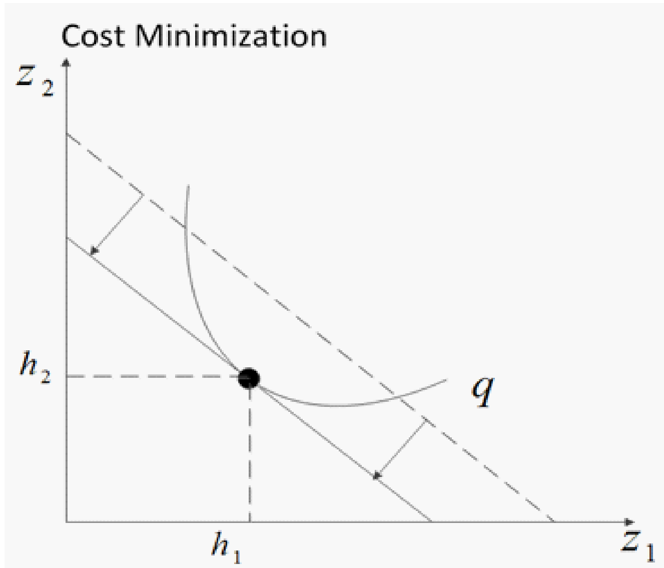



Figure 4. Cost-minimization problem.

The first order conditions are:

$$MRTS_1^2 = \frac{f_1(z)}{f_2(z)} = \frac{\alpha z_2}{\beta z_1} = \frac{w_1}{w_2}$$

and $q = z_1^{\alpha} z_2^{\beta}$.
This is a system of two equations and two unknowns ($z_1$ and $z_2$) that can be solved for the conditional factor demand functions $h_1$ and $h_2$. From the MRTS we have $z_2 = \frac{\beta}{\alpha}\frac{w_1}{w_2} z_1$, which we can replace into the constraint as follows

$$q = z_1^{\alpha} z_2^{\beta} = z_1^{\alpha} \left( \frac{\beta}{\alpha}\frac{w_1}{w_2} z_1 \right)^{\beta} = z_1^{\alpha+\beta} \left( \frac{\beta}{\alpha}\frac{w_1}{w_2} \right)^{\beta}$$

and rearranging

$$z_1^{\alpha+\beta} = q \left( \frac{\alpha w_2}{\beta w_1} \right)^{\beta}$$

which allows us to obtain the conditional factor demand correspondence $z_1(w, q)$ for input 1,

$$z_1 = h_1(w_1,\, w_2,\, q) = \left[ \frac{\alpha w_2}{\beta w_1} \right]^{\beta/(\alpha+\beta)} q^{1/(\alpha+\beta)}$$

Now replacing $z_1 = h_1(w_1,\, w_2,\, q)$ into $z_2 = \frac{\beta}{\alpha}\frac{w_1}{w_2} z_1$ we obtain the conditional factor demand correspondence $z_2(w, q)$ for input 2,

$$z_2 = h_2(w_1,\, w_2,\, q) = \left[ \frac{\beta w_1}{\alpha w_2} \right]^{\alpha/(\alpha+\beta)} q^{1/(\alpha+\beta)}$$

Using these values we can find the cost function as:

$$C(w,\, q) = w_1 h_1(w_1,\, w_2,\, q) + w_2 h_2(w_1,\, w_2,\, q)$$

that is,

$$\begin{aligned}
C(w,\, q) &= \underbrace{w_1 \left[\frac{\alpha w_2}{\beta w_1}\right]^{\beta/(\alpha+\beta)} q^{1/(\alpha+\beta)}}_{h_1(w_1, w_2, q)} + \underbrace{w_2 \left[\frac{\beta w_1}{\alpha w_2}\right]^{\alpha/(\alpha+\beta)} q^{1/(\alpha+\beta)}}_{h_2(w_1, w_2, q)} \\
&= \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\alpha+\beta}} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\alpha}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\alpha}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}} \\
&= w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\alpha}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}} \underbrace{\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right]}_{\text{Set to be } K}
\end{aligned}$$

Let $\theta = \frac{\beta}{(\alpha+\beta)}$ and $K = \left[\frac{\alpha}{\beta}\right]^{\theta} \left[\frac{\beta}{\alpha}\right]^{1-\theta}$. We can rewrite the function as:

$$C(w,\, q) = w_1^{1-\theta} w_2^{\theta} q^{\frac{1}{\alpha+\beta}} K$$

since $1 - \theta = \frac{(\alpha+\beta)-\beta}{(\alpha+\beta)} = \frac{\alpha}{(\alpha+\beta)}$.

*Profit function.* In order to find the supply correspondence and the profit function, we have to solve the profit maximization problem as follows:

$$\max_{q \geq 0} \;\; \pi(q) = p \cdot q - C(w,\, q)$$

$$\max_{q} \pi(q) = p \cdot q - K q^{1/(\alpha+\beta)} w_1^{1-\theta} w_2^{\theta}$$

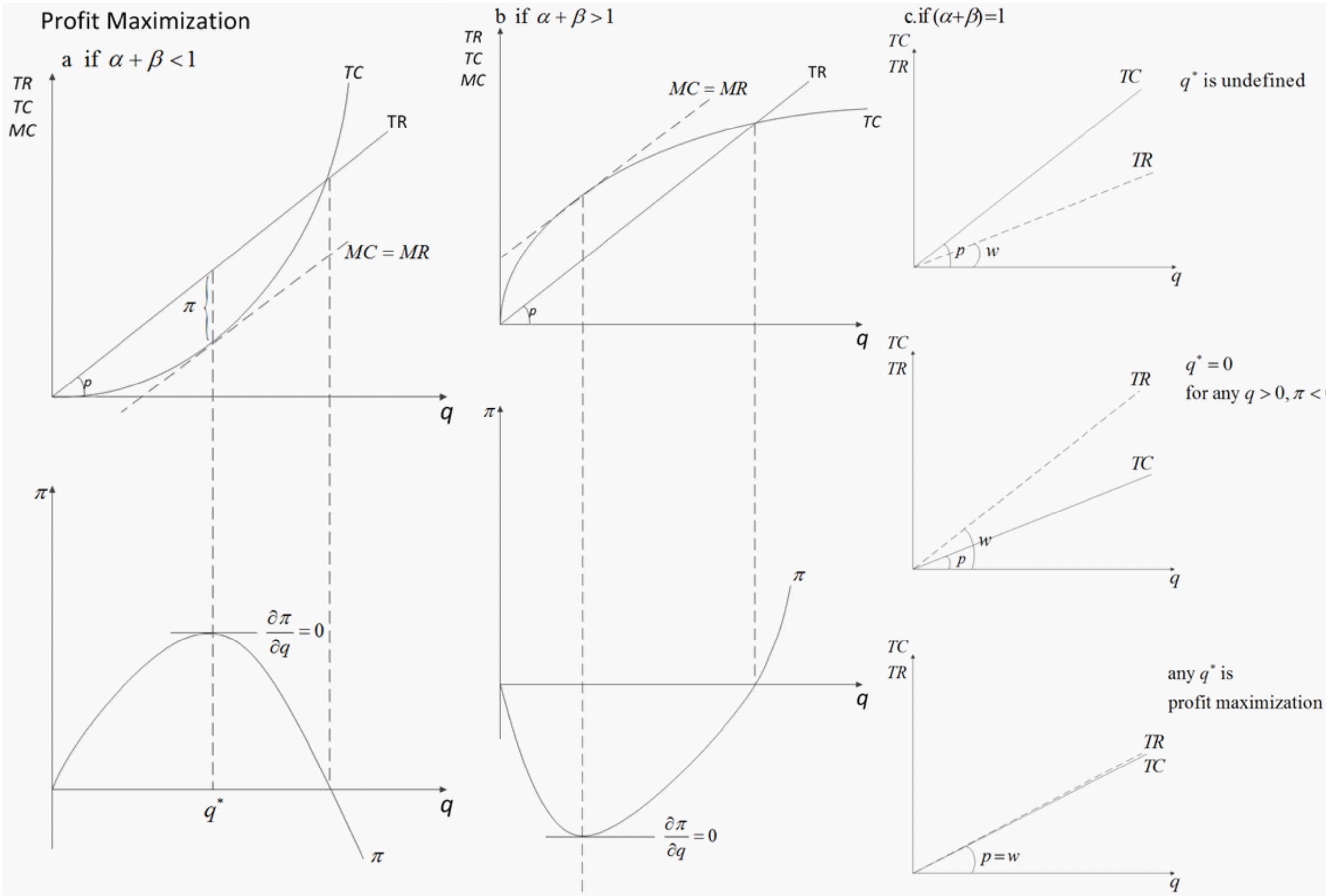
The first order conditions are:

$$\frac{\partial \pi(q)}{\partial q} = p - \left(\frac{1}{\alpha+\beta}\right) K q^{\frac{1}{\alpha+\beta}-1} w_1^{1-\theta} w_2^{\theta} \tag{1}$$

And the second order derivative must satisfy:

$$\frac{\partial^2 \pi(q)}{\partial q^2} = -\left(\frac{1}{\alpha+\beta} - 1\right)\left(\frac{1}{\alpha+\beta}\right) K q^{\frac{1}{\alpha+\beta}-2} w_1^{1-\theta} w_2^{\theta} < 0$$

Note that when $(\alpha+\beta) < 1$ the above second order condition is satisfied. Intuitively, this condition holds when the function shows *decreasing* returns to scale. Hence, only under decreasing returns to scale for this Cobb-Douglas production function we can find supply correspondences that maximize the profits and a supply function that is nondecreasing in price (satisfying the law of supply). [For a graphical illustration, see the figures separately depicting the case of $\alpha+\beta<1$, $\alpha+\beta>1$ and $\alpha+\beta=1$ below.]




Solving for $q$ from (1) we have:

$$p-\left(\frac{1}{\alpha+\beta}\right)Kq^{\frac{1}{\alpha+\beta}-1}w_1^{1-\theta}w_2^{\theta}=0$$

$$q(w,\,p)=\left[\frac{\alpha+\beta}{K}\cdot\frac{p}{w_1^{1-\theta}w_2^{\theta}}\right]^{\frac{\alpha+\beta}{1-\alpha-\beta}}$$

and now using this expression we can obtain the unconditional factor demand for factors $z_1$and $z_2$ (note that they depend on input and output prices, but are independent on total output, unlike conditional factor demand correspondences).

$$\begin{aligned}
z_1(w_1,\,w_2,\,p) &= \left[\frac{\alpha w_2}{\beta w_1}\right]^{\beta/(\alpha+\beta)} q^{1/(\alpha+\beta)} \\
&= \left[\frac{\alpha w_2}{\beta w_1}\right]^{\beta/(\alpha+\beta)} \left[\left[\frac{\alpha+\beta}{K}\cdot\frac{p}{w_1^{1-\theta}w_2^{\theta}}\right]^{\frac{\alpha+\beta}{1-\alpha-\beta}}\right]^{1/(\alpha+\beta)}
\end{aligned}$$

$$\begin{aligned}
z_2(w_1,\,w_2,\,p) &= \left[\frac{\beta w_1}{\alpha w_2}\right]^{\alpha/(\alpha+\beta)} q^{1/(\alpha+\beta)} \\
&= \left[\frac{\beta w_1}{\alpha w_2}\right]^{\alpha/(\alpha+\beta)} \left[\left[\frac{\alpha+\beta}{K}\cdot\frac{p}{w_1^{1-\theta}w_2^{\theta}}\right]^{\frac{\alpha+\beta}{1-\alpha-\beta}}\right]^{1/(\alpha+\beta)}
\end{aligned}$$

And rearranging,

$$z_1(w_1,\, w_2,\, p) = \left[\frac{\alpha w_2}{\beta w_1}\right]^{\beta/(\alpha+\beta)} \left[\frac{\alpha+\beta}{K}\cdot\frac{p}{w_1^{1-\theta}w_2^{\theta}}\right]^{\frac{1}{1-\alpha-\beta}}$$

$$z_2(w_1,\, w_2,\, p) = \left[\frac{\beta w_1}{\alpha w_2}\right]^{\alpha/(\alpha+\beta)} \left[\frac{\alpha+\beta}{K}\cdot\frac{p}{w_1^{1-\theta}w_2^{\theta}}\right]^{\frac{1}{1-\alpha-\beta}}$$

Finally, we can calculate the profit function $\pi(q) = p\cdot q - w_1 z_1 - w_2 z_2$ as:

$$\begin{aligned}\pi(q) \quad = \quad & p\cdot\left[\frac{\alpha+\beta}{K}\cdot\frac{p}{w_1^{1-\theta}w_2^{\theta}}\right]^{\frac{\alpha+\beta}{1-\alpha-\beta}} \\ & - w_1 \underbrace{\left[\frac{\alpha w_2}{\beta w_1}\right]^{\beta/(\alpha+\beta)} \left[\frac{\alpha+\beta}{K}\cdot\frac{p}{w_1^{1-\theta}w_2^{\theta}}\right]^{\frac{1}{1-\alpha-\beta}}}_{z_1(w_1,w_2,q)} - w_2 \underbrace{\left[\frac{\beta w_1}{\alpha w_2}\right]^{\alpha/(\alpha+\beta)} \left[\frac{\alpha+\beta}{K}\cdot\frac{p}{w_1^{1-\theta}w_2^{\theta}}\right]^{\frac{1}{1-\alpha-\beta}}}_{z_2(w_1,w_2,q)}\end{aligned}$$