EconS 501 - Micro Theory I
Recitation #4b - Demand theory (Applications)[1]

1. **Exercise 3.I.7 MWG:** There are three commodities (i.e., L=3) of which the third is a numeraire (let $p_3 = 1$) the Walrasian demand function for each good $x(p, w)$ is

$$x_1(p, w) = a + bp_1 + cp_2$$

$$x_2(p, w) = d + ep_1 + gp_2$$

a) **Give the parameter restrictions implied by utility maximization.**

- Intuitively, note that:

1. $b \leq 0$ for ULD (i.e., $\triangle p \cdot \triangle x \leq 0$) to be satisfied ($\uparrow p_1 \Rightarrow \downarrow x_1$)
2. $g \leq 0$ for ULD to be satisfied ($\uparrow p_2 \Rightarrow \downarrow x_2$)
3. What about the sign of c (or e)?
   (a) if $c > 0$, then $p_2 \Rightarrow \uparrow x_1$ (i.e., x_1 and x_2 are substitutes)
   (b) if $c < 0$, then $p_2 \Rightarrow \downarrow x_1$ (i.e., x_1 and x_2 are complements)

Let's analyse this more formally. By applying Walras' law and homogeneity of degree zero, we can obtain the demand functions for all three goods defined over the domain $\{(p, w) \in \mathbb{R}^3 \times \mathbb{R} : p \gg 0\}$. Thus, we can obtain the 3×3 Slutsky matrix as well from the demand functions. In particular, since there are no income effects (by looking at the Walrasian demand, we can see that $\frac{\partial x_k(p,w)}{\partial w} = 0$ for any good k), we can express the Slutsky matrix as follows (where each entry in the Slutsky matrix implies that substitution and total effect coincide):

$$S(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \frac{\partial x_1(p,w)}{\partial p_2} & \frac{\partial x_1(p,w)}{\partial p_3} \\ \frac{\partial x_2(p,w)}{\partial p_1} & \frac{\partial x_2(p,w)}{\partial p_2} & \frac{\partial x_2(p,w)}{\partial p_3} \\ \frac{\partial x_3(p,w)}{\partial p_1} & \frac{\partial x_3(p,w)}{\partial p_2} & \frac{\partial x_3(p,w)}{\partial p_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \frac{\partial x_1(p,w)}{\partial p_2} & 0 \\ \frac{\partial x_2(p,w)}{\partial p_1} & \frac{\partial x_2(p,w)}{\partial p_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

[Recall that we can delete the third column and third row because all their elements are zero and the 3rd principal minor is also zero.] The 2×2 submatrix of the Slutsky matrix that is obtained by deleting the bottom row and the right-hand column is:

$$S(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \frac{\partial x_1(p,w)}{\partial p_2} \\ \frac{\partial x_2(p,w)}{\partial p_1} & \frac{\partial x_2(p,w)}{\partial p_2} \end{bmatrix} = \begin{bmatrix} b & c \\ e & g \end{bmatrix}$$

[1] Felix Munoz-Garcia, School of Economic Sciences, Washington State University, 103G Hulbert Hall, Pullman, WA. Tel. 509-335-8402. Email: fmunoz@wsu.edu.

The original 3×3 Slutsky matrix is symmetric if and only if this 2×2 matrix is symmetric.[2] Moreover, just as in the proof of Theorem M.D.4(iii), we can show that the 3×3 Slutsky matrix is negative semidefinite on $\mathbb{R}^3$ if and only if the 2×2 matrix is negative semidefinite. In particular this matrix is symmetric if $c = e$, and negative semidefinite if the elements along the main diagonal satisfy $b \leq 0$, $g \leq 0$, and its determinent, $bg - c^2$, is positive.

b) **Estimate the Equivalent Variation for a change of prices from $(p_1, p_2) = (1, 1)$ to $(\overline{p}_1, \overline{p}_2) = (2, 2)$. Verify that without appropriate symmetry, there is no path independence. Assume independence for the rest of the exercise.**

Let p be any price vector and u, u' be any two utility levels. By duality (see ,for instance, (3.E.4) in MWG) we have:

$$h_l(p, u) = x_l(p, e(p, u)) \text{ and } h_l(p, u') = x_l(p, e(p, u')) \text{ for every good } l = 1, 2$$

also, since the walrasian demands in this exercise $x_l(\bullet)$ do not depend on wealth, we can write

$$x_l(p, e(p, u)) = x_l(p, e(p, u'))$$

then we have $h_l(p, u) = h_l(p, u')$. Hence, the hicksiand demands $h_l(p, u)$ do not depend on utility level and they are the same as the $x_l(p, w)$ in this exercise.

Let us now examine how the path of price increases might affect the size of the equivalent variation (EV):

First path Let us first assume that prices change following the path $(1, 1) \to (2, 1) \to (2, 2)$: First, we must find the EV of increasing p_1 from $p_1 = 1$ to $p_1 = 2$. Second, we must find the EV of increasing p_2 from $p_2 = 1$ to $p_2 = 2$, as follows;

$$EV = \int_1^2 h^1(p_1,\, 1,\, u)dp_1 + \int_1^2 h^2(2,\, p_2,\, u)dp_2$$

And since Hicksian and Walrasian demands coincide in this exercise,

$$EV = \int_1^2 x^1(p_1,\, 1,\, w)dp_1 + \int_1^2 x^2(2,\, p_2,\, w)dp_2$$

Plugging the expression the Walrasian demand functions,

$$EV = \int_1^2 (a + bp_1 + c)dp_1 + \int_1^2 (d + 2e + gp_2)dp_2$$

[2]Note that if the 2×2 matrix is symetric, then adding a new column of zeros at the right hand side and a row of zeros at the bottom row still yields a symmetric matrix (indeed, all elements above and below the main diagonal coincide).

where we fixed $p_2 = 1$ in the first term (where only p_1 changes) and $p_1 = 2$ in the second term (where only p_2 changes). Integrating,

$$EV = \left(a + \frac{3}{2}b + c\right) + \left(d + 2e + \frac{3}{2}g\right) \tag{1}$$

Second path Let us now consider that prices change following the path $(1,1) \to (1,2) \to (2,2)$. Note that using this path for increasing prices, we first raise p_2 from $p_2 = 1$ to $p_2 = 2$, and then we raise p_1 from $p_1 = 1$ to $p_1 = 2$. Hence, in order to find the EV of these price changes, we must first find the EV of increasing p_2 (from $p_2 = 1$ to $p_2 = 2$), and second, for a fixed level of $p_2 = 2$, we must find the EV of increasing p_1 (from $p_1 = 1$ to $p_1 = 2$), as follows;

$$EV = \int_1^2 h^2(1,\, p_2,\, u)dp_2 + \int_1^2 h^1(p_1,\, 2,\, u)dp_1$$

And since Hicksian and Walrasian demands coincide in this exercise,

$$EV = \int_1^2 x^2(1,\, p_2,\, w)dp_2 + \int_1^2 x^1(p_1,\, 2,\, w)dp_1$$

Plugging the Walrasian demand function, yields

$$EV = \int_1^2 (d + e + gp_2)dp_2 + \int_1^2 (a + bp_1 + 2c)dp_1$$

where we fixed $p_1 = 1$ in the first term (where only p_2 changes)and $p_2 = 2$ in the second term (where only p_1 changes). Integrating,

$$EV = \left(d + e + \frac{3}{2}g\right) + \left(a + \frac{3}{2}b + 2c\right) \tag{2}$$

Note that the equivalent variation following the first path (expression 1) and following the second path (expression 2) coincide if and only if $c = e$ (which we required in order to have a symmetric Slustky matrix).

- Hence, when the Slustky matrix is symmetric we can guarantee that an increase in the price of the two goods is "path independent", since it yields the same EV regardless of whether p_1 or p_2 is the first to change.

c) **Let EV_1, EV_2 and EV be the equivalent variations for a change of prices from $(p_1,\, p_2) = (1,1)$ to respectively $(2,1), (1,2)$, and $(2,2)$. Compare EV with $EV_1 + EV_2$ as a function of the parameters of the problem. Interpret.**

Let us define the notation we will use in this part of the exercise.

- EV_1 measures the EV for the price change (1,1) to (2,1) - Only p_1 increases.
- EV_2 measures the EV for the price change (1,1) to (1,2) - Only p_2 increases.
- EV measures the EV for the price change (1,1) to (2,2) - Both prices increase simultaneously.

EV_1. Following a similar approach as in part (b) of the exercise, if only $p_1$increases from $p_1 = 1$ to $p_1 = 2$ (while p_2 remains at $p_2 = 1$), we obtain an equivalent variation of

$$EV_1 = \int_1^2 x^1(p^1,\, 1,\, w)dp^1 = a + \frac{3}{2}b + c$$

as depicted in figure 1.

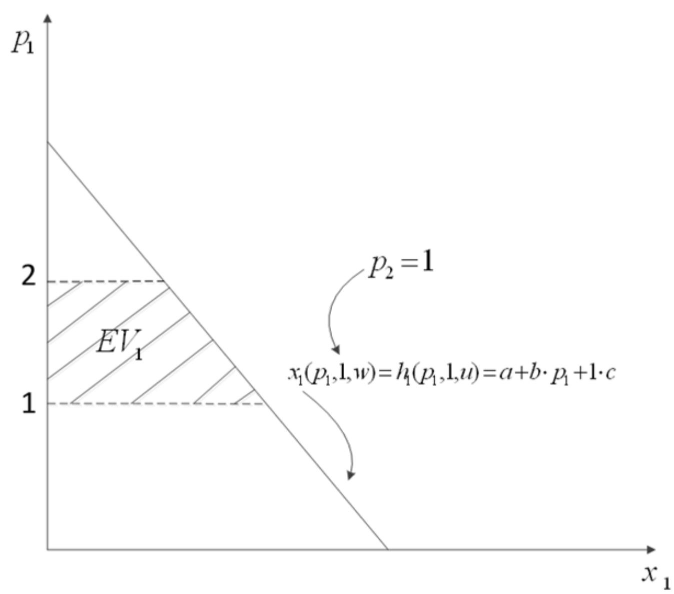
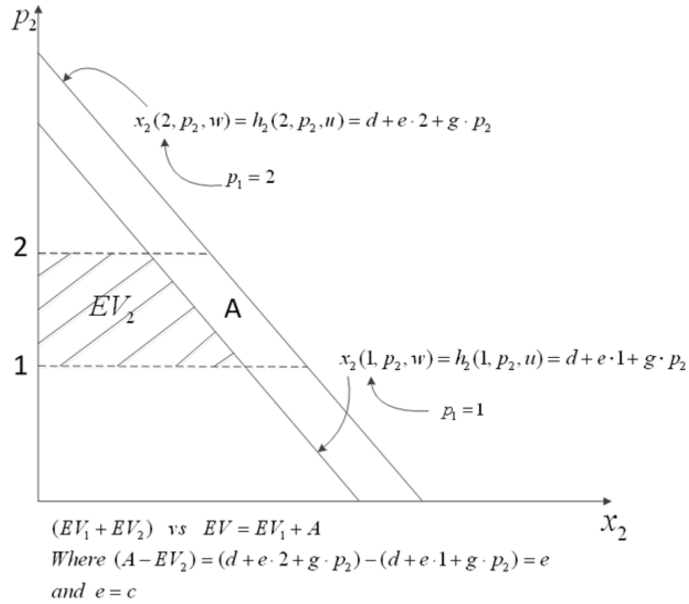


Figure 1. EV_1

EV_2. If only $p_2$increases from $p_2 = 1$ to $p_2 = 2$ (while p_1 remains at $p_1 = 1$), the equivalent variation is

$$EV_2 = \int_1^2 x^2(1,\, p^2,\, w)dp^2 = d + e + \frac{3}{2}g$$



Fgure 2. EV_2.

EV.We now want to find the equivalent variation from a simultaneous increase in the price of both goods denoted by EV in this exercise. Remember from part (b) that we can increase the price of both goods following two different paths. Let us first find the EV from increasing the price of both goods by following the first path:

$$EV = \int_1^2 x^1(p^1,\, 1,\, w)dp^1 + \int_1^2 x^2(2,\, p^2,\, w)dp^2$$

$$EV = (a + \frac{3}{2}b + c) + (d + 2e + \frac{3}{2}g)$$

Let us now find the EV by following the second path:

$$EV = \int_1^2 x^2(1,\, p^2,\, w)dp^2 + \int_1^2 x^1(p^2,\, 2,\, w)dp^1$$

$$EV = \left(d + e + \frac{3}{2}g\right) + \left(a + \frac{3}{2}b + 2c\right)$$

And in the case that the Slutsky matrix is symmetric, $c = e$, we have that the EV from increasing the price of both goods is "path independent" and takes the value:

$$EV = a + \frac{3}{2}b + 3c + d + \frac{3}{2}g$$

Difference between EV and $(EV_1{+}EV_2)$. Let us now find the difference between EV (resulting from simultaneous increasing the price of both goods) and the sum of EV_1 and EV_2.

$$EV - (EV_1 + EV_2) = \left(a + \frac{3}{2}b + 3c + d + \frac{3}{2}g\right) - \left(a + \frac{3}{2}b + 2c + d + \frac{3}{2}g\right) = c.$$

The sum $EV_1 + EV_2$ does not contain the effect on the equivalent variation due to the shift of the graph of the demand function for the second commodity when p_1 goes up to 2 (or equivalently, the shift of the graph of the demand function for the first commodity when p_2 goes up to 2). (See figures at the end of the handout, for a graphical comparison between the area $EV_1 + EV_2$ and the area EV.

d) **Suppose that the prices increases described in part (c) are due to taxes. Denote the deadweight losses for each of the three experiments by DW_1, DW_2, and DW. Compare DW with $DW_1 + DW_2$ as a function of parameters of the problem.**

DW_1 **.** We first calculate the deadweight loss if the tax affects the price of good 1 alone, DW_1, raising it from $p_1 = 1$ to $p_1 = 2$. First, note that the tax rate is \$1. Hence, since

$$x_1(2, 1, w) = a + 2b + c$$

the tax revenue from the first good is equal to $T_1 = 1 \times x_1(2, 1, w)$. (See the figure 3 representing DW_1).

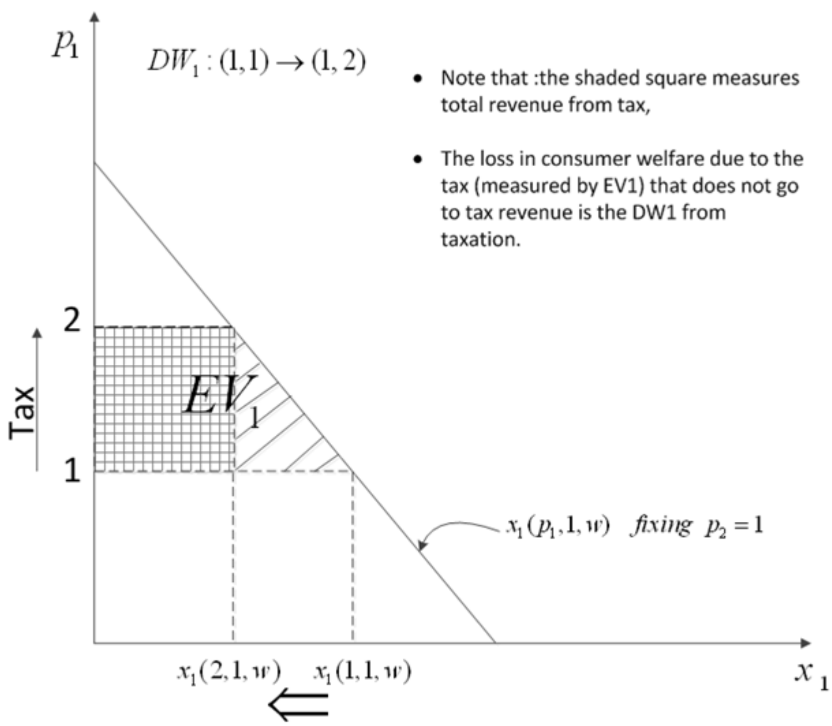


Figure 3. DW_1.

Thus, since the equivalent variation represents a welfare loss from the introdution of the tax, $-EV_1 = T_1 + DW_1$, then

$$DW_1 = T_1 - EV_1 = (a + 2b + c) - \left(a + \frac{3}{2}b + c\right) = \frac{b}{2}.$$

DW_2. We secondly calculate the deadweight loss if the tax affects the price of good 2 alone, DW_2, raising it from $p_2 = 1$ to $p_2 = 2$. First, note that the tax rate is \$1. Hence, since $-EV_2 = T_2 + DW_2$, then

$$x_2(1, 2, w) = d + e + 2g$$

the tax revenue from the second good is equal to $T_2 = 1 \times x_2(1, 2, w)$. (See the figure 4 representing DW_2).

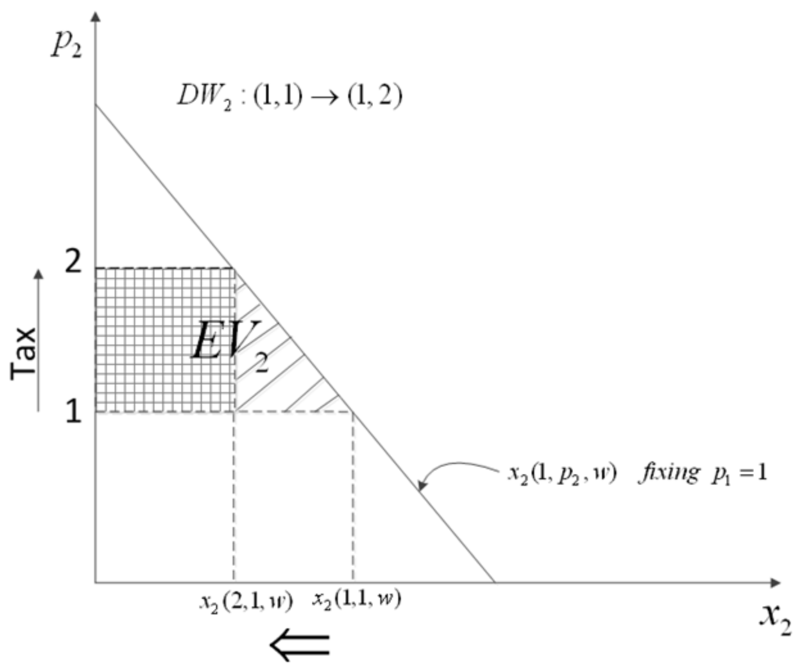


Figure 4. DW_2.

Thus, since the equivalent variation represents a welfare loss from the introdution of the tax, $-EV_2 = T_2 + DW_2$, then

$$DW_2 = T_2 - EV_2 = (d + e + 2g) - \left(d + e + \frac{3}{2}g\right) = \frac{g}{2}.$$

DW. Third, we now find the deadweight loss from a tax that affects both the price of good 1 and the price of good 2. First, note that since $x_1(2, 2, w) = a + 2b + 2c$, and $x_2(2, 2, w) = d + 2e + 2g$, the tax revenue from taxing both commodities is equal to:

$$T = 1 \times (a + 2b + 2c) + 1 \times (d + 2e + 2g) = a + 2b + 4c + d + 2g$$

Then, since $-EV = T + DW$, the deadweight loss in this case is $DW = T - EV$

$$DW = T - EV = (a + 2b + 4c + d + 2g) - \left(a + \frac{3}{2}b + 3c + d + \frac{3}{2}g\right) = \frac{b}{2} + c + \frac{g}{2}$$

Let us finally examine the difference between calculating the deadweight loss of the tax that simultaneously affects the price of both commodities, DW, and the sum of the deadweight loss of the tax affecting the price of each commodity separately, *i.e.*, $DW_1 + DW_2$. It is easy to check that

$$DW - (DW_1 + DW_2) = c$$

e) **Suppose the initial tax situation has prices $(p_1, p_2) = (1, 1)$. The government wants to raise a fixed (small) amount of revenue R through commodity taxes. Call t_1 and t_2 the tax rates for the two commodities. Determine the optimal tax rates as a function of the parameters of demand if the optimality criterion is the minimization of the deadweight loss.**

The government's problem is:

$$\min_{(t_1,t_2)} \quad DW(t_1, t_2)$$

$$\text{subject to } \sum_{l=1}^{2} h_l(1 + t_1, 1 + t_2, u) \times t_l \geq R$$

where $DW(t_1, t_2) = TR(t_1, t_2) - EV(t_1, t_2)$ is,

$$DW(t_1, t_2) = \underbrace{\sum_{l=1}^{2} h_l(1 + t_1, 1 + t_2, u)t_l}_{TR(t_1,t_2)} - \overbrace{[e(1 + t_1, 1 + t_2, u) - e(1, 1, u)]}^{EV(t_1,t_2)}$$

where $TR(t_1, t_2)$ represents the total tax revenue from setting a sales tax $t_1(t_2)$ on good 1 (good 2, repectively), while $EV(t_1, t_2)$ denotes the equivalent variation of experiencing an increase in both goods prices from $(p_1, p_2) = (1, 1)$ to $(p_1, p_2) = (1+t_1, 1+t_2)$ ofter the taxes are introduced.
Setting up the Lagrangian

$$L(t_1, t_2, \lambda) = DW(t_1, t_2) + \lambda(R - TR(t_1, t_2))$$

Then the first order condition with respect to t_l is:

$$\frac{\partial DW(t_1, t_2)}{\partial t_l} - \lambda \frac{\partial TR(t_1, t_2)}{\partial t_l} = 0 \;\text{ for every good } l = \{1, 2\} \tag{3}$$

Note that the term in the left-hand side can be rewritten as

$$\frac{\partial DW(t_1,t_2)}{\partial t_l} = \sum_{k=1}^{2} \frac{\partial h_k(1+t_1,1+t_2,u)}{\partial t_l} t_k - \left[\frac{\partial e(1+t_1,1+t_2,u)}{\partial t_l} - h_l(1+t_1,1+t_2,u) \right]$$

since $\frac{\partial e(1+t_1,1+t_2,u)}{\partial t_l} = h_l(1+t_1,1+t_2,u)$ in the last term. Then, $\frac{\partial DW(t_1,t_2)}{\partial t_l} = \sum_{k=1}^{2} \frac{\partial h_k(1+t_1,1+t_2,u)}{\partial t_l} t_k$, on the other hand, the second term of expression (3) can be rewitten as

$$\frac{\partial TR(t_1,t_2)}{\partial t_l} = h_l(1+t_1,1+t_2,u) + \sum_{k=1}^{2} \frac{\partial h_k(1+t_1,1+t_2,u)}{\partial t_l} t_k$$

Hence, the above first order condition can be written as:

$$\sum_{k=1}^{2} \frac{\partial h_k(1+t_1,1+t_2,u)}{\partial t_l} t_k - \lambda \left[h_l(1+t_1,1+t_2,u) + \sum_{k=1}^{2} \frac{\partial h_k(1+t_1,1+t_2,u)}{\partial t_l} t_k \right] = 0$$

And rearranging,

$$\sum_{k=1}^{2} \frac{\partial h_k(1+t_1,1+t_2,u)}{\partial t_l} t_k (1+\lambda) - \lambda h_l(1+t_1,1+t_2,u) = 0 \text{ for all } l = 1,2.$$

From this expression and $TR = \sum_{l=1}^{2} h_l(1+t_1,1+t_2,u) \times t_l$ we obtain

$$-\lambda = \frac{bt_1 + ct_2}{a + b(1+2t_1) + c(1+2t_2)} = \frac{ct_1 + gt_2}{a + c(1+2t_1) + g(1+2t_2)}$$

and

$$R = [a + b(1+t_1) + c(1+t_2)]\, t_1 + [d + c(1+t_1) + g(1+t_2)]\, t_2$$

Therefore, any combination of tax rates (t_1, t_2) that satisfies the previous condition minimizes the total deadweight loss of taxation, DW, and allows the tax authority to reach a minimal tax revenue of TR dollars. For instance, if $R = \$4$, and the parameters in the demand function are $a = c = d = 1$ and $b = g = -1$ the above expression becomes

$$\$4 = [1 - (1+t_1) + (1+t_2)]\, t_1 + [1 + (1+t_1) - (1+t_2)]\, t_2 \qquad (4)$$

which only depends on t_1 and t_2. Hence, any $(t_1, t_2)-$combinations satisfying equation (4) allow the regulator reach a tax revenue of $R = \$4$, while minimizing the deadweight less of taxation.