Exercise 1. Prove that Shephard's lemma is implied by Roy's identity. [*Hint*: Assume that we are at an optimum.]

*Answer***.** Since the identity $v\big(p,e(p,u)\big)=u$ holds for all *p*, differentiation with respect to *p* yields

$$\nabla_p v\big(p,e(p,u)\big)+\frac{\partial v\big(p,e(p,u)\big)}{\partial w}\nabla_p e(p,u)=0. \qquad ①$$

where $w\equiv e(p,u)$, and we differentiate the first argument $v(p,\ \cdot\ )$ and the second argument $v\big(\ \cdot\ ,e(p,u)\big)$. We apply the chain rule when differentiating the second term.

By Roy's identity, we have

$$x_l(\bar{p},\bar{w})=-\frac{\dfrac{\partial v\big(p,e(p,u)\big)}{\partial p_l}}{\dfrac{\partial v\big(p,e(p,u)\big)}{\partial w}}$$

Or rearranging,

$$-\frac{\partial v\big(p,e(p,u)\big)}{\partial w}\cdot x_l(\bar{p},\bar{w})=\frac{\partial v\big(p,e(p,u)\big)}{\partial p_l} \qquad ②$$

Substituting ② into the first term of ①,

$$-\frac{\partial v\big(p,e(p,u)\big)}{\partial w}\cdot x_l(\bar{p},\bar{w})+\frac{\partial v\big(p,e(p,u)\big)}{\partial w}\nabla_p e(p,u)=0$$

$$\Rightarrow \frac{\partial v\big(p,e(p,u)\big)}{\partial w}\Big[-x_l(\bar{p},\bar{w})+\nabla_p e(p,u)\Big]=0$$

Finally, by $\dfrac{\partial v\big(p,e(p,u)\big)}{\partial w}>0$ (Maximal utility from the UMP is increasing in income.) and $h(p,u)=x\big(p,e(p,u)\big)$ (considered at an optimum) we obtain

$$h(p,u)=\nabla_p e(p,u).$$

Exercise 2. Verify for the case of a Cobb-Douglas utility function, $u(x)=x_1^{\alpha}x_2^{1-\alpha}$, that all of the propositions in Section 3.G hold.

Answer. Recall that, from Example 3.D.1, the Walrasian demand for the Cobb-Douglas utility function are

$$x_1(p,w)=\frac{\alpha}{p_1}w \quad \text{and} \quad x_2(p,w)=\frac{1-\alpha}{p_2}w.$$

Hence, its derivative with respect to wealth is

$$D_w x(p,w)=\begin{bmatrix}\frac{\alpha}{p_1}\\ \frac{1-\alpha}{p_2}\end{bmatrix},$$

(As a remark, note that these derivatives are positive, thus confirming that both goods 1 and 2 are normal rather than inferior.)

The derivative with respect to prices is

$$D_p x(p,w)=\begin{bmatrix}-\frac{\alpha w}{p_1^2} & 0\\ 0 & -\frac{(1-\alpha)w}{p_2^2}\end{bmatrix},$$

which indicates that the demand for every good k decreases in its own price, but is unaffected by the price of the other good.
From Example 3.E.1, the Hicksian demand in the case of a Cobb-Douglas utility function is

$$h_1(p,u)=\left[\frac{\alpha}{1-\alpha}\cdot\frac{p_2}{p_1}\right]^{1-\alpha}\cdot u$$
$$h_2(p,u)=\left[\frac{1-\alpha}{\alpha}\cdot\frac{p_1}{p_2}\right]^{\alpha}\cdot u$$

And the associated expenditure function $e(p,u)=p\cdot h(p,u)$ yields

$$e(p,u)=p_1\cdot h_1(p,u)+p_2\cdot h_2(p,u)$$
$$\Rightarrow e(p,u)=p_1\cdot\left[\frac{\alpha}{1-\alpha}\cdot\frac{p_2}{p_1}\right]^{1-\alpha}\cdot u+p_2\left[\frac{1-\alpha}{\alpha}\cdot\frac{p_1}{p_2}\right]^{\alpha}\cdot u$$
$$\Rightarrow e(p,u)=\frac{\alpha^{1-\alpha}}{(1-\alpha)^{1-\alpha}}\cdot\frac{p_2^{1-\alpha}}{p_1^{1-\alpha}}\cdot p_1\cdot u+\frac{(1-\alpha)^{\alpha}}{\alpha^{\alpha}}\cdot\frac{p_1^{\alpha}}{p_2^{\alpha}}\cdot p_2\cdot u$$
$$\Rightarrow e(p,u)=\frac{\alpha(1-\alpha)^{\alpha-1}p_2^{1-\alpha}p_2^{\alpha-1}+(1-\alpha)^{\alpha}p_1^{\alpha}p_1^{-\alpha}}{\alpha^{\alpha}p_1^{-\alpha}\cdot p_2^{\alpha-1}}\cdot u$$
$$\Rightarrow e(p,u)=\frac{\alpha(1-\alpha)^{\alpha-1}+(1-\alpha)^{\alpha}}{\alpha^{\alpha}p_1^{-\alpha}\cdot p_2^{\alpha-1}}\cdot u=\frac{(1-\alpha)^{\alpha-1}(\alpha+1-\alpha)}{\alpha^{\alpha}p_1^{-\alpha}\cdot p_2^{\alpha-1}}\cdot u$$
$$\Rightarrow e(p,u)=\left[\frac{(1-\alpha)^{\alpha-1}}{\alpha^{\alpha}}\cdot\frac{p_1^{\alpha}}{p_2^{\alpha-1}}\right]\cdot u$$

We now seek to confirm that the derivative of $e(p.u)$ with respect to prices, $\nabla_p e(p,u)$, yields the

Hicksian demands we defined above (that is, we seek to confirm Shephard's lemma) . Indeed,

$$\frac{\partial e(p,u)}{\partial p_1} = u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha}\cdot\frac{\alpha}{p_1} = h_1(p,u)$$

and

$$\frac{\partial e(p,u)}{\partial p_2} = u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha}\cdot\frac{1-\alpha}{p_2} = h_2(p,u)$$

Or, more compactly, in matrix notation,

$$\nabla e(p,u) = u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha}\begin{bmatrix}\frac{\alpha}{p_1}\\ \frac{(1-\alpha)}{p_2}\end{bmatrix},$$

In addition, we seek to test the property $D_p^2 e(p,u) = D_p h(p,u)$. Differentiating our above result with respect to p yields

$$D_p^2 e(p,u) = D_p h(p,u) = u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha}\begin{bmatrix}-\frac{\alpha(1-\alpha)}{p_1^2} & \frac{\alpha(1-\alpha)}{p_1 p_2}\\ \frac{\alpha(1-\alpha)}{p_1 p_2} & -\frac{\alpha(1-\alpha)}{p_2^2}\end{bmatrix}$$

In addition, $D_p h(p,u)$ is a negative semidefinite matrix since the principal minors satisfy

$$-\frac{\alpha(1-\alpha)}{p_1^2} < 0 \text{ and } \det\begin{bmatrix}-\frac{\alpha(1-\alpha)}{p_1^2} & \frac{\alpha(1-\alpha)}{p_1 p_2}\\ \frac{\alpha(1-\alpha)}{p_1 p_2} & -\frac{\alpha(1-\alpha)}{p_2^2}\end{bmatrix} = 0$$

Moreover, $D_p h(p,u)\cdot p = 0$ given that

$$u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha}\begin{bmatrix} -\frac{\alpha(1-\alpha)}{p_1^2} & \frac{\alpha(1-\alpha)}{p_1 p_2} \\ \frac{\alpha(1-\alpha)}{p_1 p_2} & -\frac{\alpha(1-\alpha)}{p_2^2} \end{bmatrix}\cdot\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$=u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha}\begin{bmatrix} -\frac{\alpha(1-\alpha)}{p_1^2}\cdot p_1+\frac{\alpha(1-\alpha)}{p_1 p_2}\cdot p_2 \\ \frac{\alpha(1-\alpha)}{p_1 p_2}\cdot p_1-\frac{\alpha(1-\alpha)}{p_2^2}\cdot p_2 \end{bmatrix}$$

$$=u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha}\begin{bmatrix} -\frac{\alpha(1-\alpha)}{p_1}+\frac{\alpha(1-\alpha)}{p_1} \\ \frac{\alpha(1-\alpha)}{p_2}-\frac{\alpha(1-\alpha)}{p_2} \end{bmatrix}$$

$$=u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha}\begin{bmatrix} 0 \\ 0 \end{bmatrix}=0$$

(Recall from our class discussion that this property entails that not all goods are net substitutes, nor all goods are net complements.)

The last property we seek to check is that Roy's identity holds. That is, if we differentiate the indirect utility function $v(p,w)$ with respect to p_l and w, the negative of the ratio of derivative coincides with the Walrasian demand of a Cobb-Douglas utility function, that is,

$$-\frac{\frac{\partial v(p,u)}{\partial p_l}}{\frac{\partial v(p,u)}{\partial w}}=x_l(p,w)$$

In order to show this result, first note that the indirect utility function for $u(x)=x_1^{\alpha}x_2^{1-\alpha}$ is

$$v(p,w)=\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}w.$$

Differentiating $v(p,w)$ with respect to p_1, yields

$$\frac{\partial v(p,w)}{\partial p_1}=-\alpha^{1+\alpha}p_1^{-(1+\alpha)}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}w$$

And differentiating $v(p,w)$ with respect to *w*, we obtain

$$\frac{\partial v(p,w)}{\partial w}=\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}$$

Hence, the ratio of these derivatives can be simplified as follows,

$$-\frac{\frac{\partial v(p,u)}{\partial p_l}}{\frac{\partial v(p,u)}{\partial w}} = -\frac{-\alpha^{1+\alpha} p_1^{-(1+\alpha)} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} w}{\left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}} = \frac{\alpha}{p_1} w = x_l(p,w)$$

A similar argument applies for good 2.

Exercise 3. A utility function $u(x)$ is *additively separable* if it has the form

$$u(x) = \sum_{\ell} u_{\ell}(x_{\ell}).$$

For instance, in the context of three goods, an additively separable function would be $u(x) = u_1(x_1) + u_2(x_2) + u_3(x_3)$, where function $u_{\ell}(x_{\ell})$ can be linear or nonlinear in the units of good ℓ, x_{ℓ}.[1] In words, this type of utility function indicates that the consumer only cares about the number of units of each good, but does not find interactions between the units of good *k* and the utility he derives from good *j*. Show that the induced ordering on any group of commodities is independent of whatever fixed values we attach to the remaining ones.

Answer: Define the set of goods $S = \{1, \ldots, L\}$ and let *T* be a subset of the goods in the list *S*. For instance, if the set of goods is $S = \{1, \ldots, 5\}$, subset T could include the first three goods, that is, $T = \{1, 2, 3\}$. The commodity vectors for those goods in *T* are represented by $z_1 = \{z_{\ell}\}_{\ell \in T} \in \mathbb{R}_+^{\#T}$, and similarly for commodity vectors of those goods outside *T*, which are represented by $z_2 = \{z_{\ell}\}_{\ell \notin T} \in \mathbb{R}_+^{L-\#T}$. ℝContinuing with our above example, $z_1 \in \mathbb{R}_+^3$ since the cardinality of set *T* is 3 (it contains three goods), and $z_2 \in \mathbb{R}_+^2$ given that there are two remaining goods (for a total of 5). For instance, we could have $z_1 = \{15, 6, 8\}$, indicating 15 units of good 1, 6 units of good 2, and 8 of good 3; and $z_2 = \{11, 7\}$, representing 11 units of good 4 and 7 of good 5.

We shall prove that for every $z_1 \in \mathbb{R}_+^{\#T}$, $z_1' \in \mathbb{R}_+^{\#T}$, $z_2 \in \mathbb{R}_+^{L-\#T}$, and $z_2' \in \mathbb{R}_+^{L-\#T}$,

$$(z_1, z_2) \succsim (z_1', z_2) \text{ if and only if } (z_1, z_2') \succsim (z_1', z_2').$$

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[1] For example, $u_{\ell}(x_{\ell})$ could be $u_{\ell}(x_{\ell}) = ax_{\ell}$ or $u_{\ell}(x_{\ell}) = ax_{\ell}^2$ where *a*>0; or more generally, functions of the form $u_{\ell}(x_{\ell}) = ax_{\ell}^{\beta}$, where $a, \beta \in \mathbb{R}$.

That is, for a given commodity vector z_2, the consumer compares z_1 against z'_1. In other words, his preference for z_1 over z'_1 (where he only considers T goods) is unaffected by the specific commodity vector z_2 that he consumes of all other goods.

In fact, since this preference relation $\succsim$ is represented with the above additively separable utility function, $(z_1, z_2) \succsim (z_1', z_2)$ is equivalent to

$$\sum_{\ell \in T} u_\ell(z_\ell) + \sum_{\ell \notin T} u_\ell(z_\ell) \geq \sum_{\ell \in T} u_\ell(z'_\ell) + \sum_{\ell \notin T} u_\ell(z_\ell).$$

which simplifies to

$$\sum_{\ell \in T} u_\ell(z_\ell) \geq \sum_{\ell \in T} u_\ell(z'_\ell).$$

Likewise, $(z_1, z_2') \succsim (z_1', z_2')$ is equivalent to

$$\sum_{\ell \in T} u_\ell(z_\ell) + \sum_{\ell \notin T} u_\ell(z'_\ell) \geq \sum_{\ell \in T} u_\ell(z'_\ell) + \sum_{\ell \notin T} u_\ell(z'_\ell).$$

which also simplifies to

$$\sum_{\ell \in T} u_\ell(z_\ell) \geq \sum_{\ell \in T} u_\ell(z'_\ell).$$

Hence, they are equivalent to each other.

b) Show now that the Walrasian and Hicksian demand functions generated by an additively separable utility function admit no inferior goods if the functions $u_\ell(\cdot)$ are strictly concave. (You can assume differentiability and interiority to answer this question.)

Answer: First, we know that the following tangency condition holds both in the UMP and in the EMP:

$$MRS_{k,l} = \frac{u'_k(x_k(p,w))}{u'_\ell(x_\ell(p,w))} = \frac{p_k}{p_\ell}$$

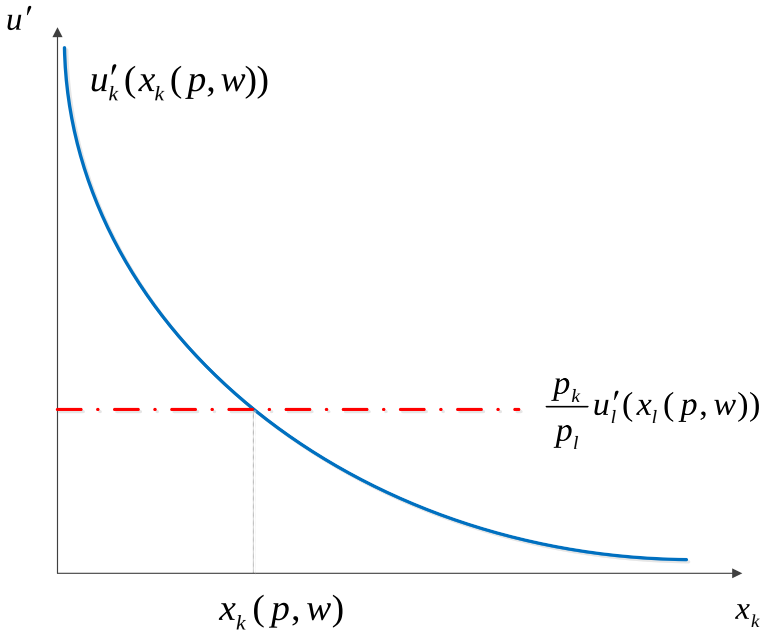
Importantly, in this context the marginal utility of good k is only function of the units of good k that the individual consumes, but is independent on the units of other goods. A similar argument applies to the marginal utility of good ℓ.[2]

Rearranging the above tangency condition, we obtain

$$u'_k(x_k(p,w)) = \frac{p_k}{p_\ell} u'_\ell(x_\ell(p,w)) \quad for\ every\ k = 1, \ldots L \quad .$$

[2] For instance, if the utility function is $u(x,y) = ax^2 + by$, the marginal utility of x is a function of x alone, and similarly for good y. However, if the utility function is *not* additively separable, for example if $u(x,y) = a(xy)^2 + by$, the marginal utility of x is a function of both x and y.

The next figure depicts the term in the left-hand side of the above expression, $u_k'(x_k(p,w))$, which is positive for all x_k (i.e., positive marginal utility), but decreasing (i.e., diminishing marginal utility). In addition, the figure also illustrates the term in the right-hand side, $\frac{p_k}{p_l}u_l'(x_l(p,w))$, which is constant in x_k.



Suppose now that the wealth level w increases and all prices remain unchanged. Then the demand for at least one good (say, good ℓ) has to increase by Walras' law. We seek to show that the demand for the remaining good k must also increase, thus implying that all goods are normal.

If the demand for good ℓ increases, its marginal utility $u_l'(x_l(p,w))$ decreases. [This follows by concavity, which intuitively represents diminishing marginal utility in the consumption of all goods.] Graphically, a decrease in $u_l'(x_l(p,w))$ implies that the line representing $\frac{p_k}{p_l}u_l'(x_l(p,w))$ shifts downwards, yielding a new crossing point to the right-hand side of the initial crossing point depicted in the above figure. As a consequence, the consumer demands a larger amount of good k, i.e., $x_k(p,w)$ increases, ultimately implying that good k is normal. Thus, all goods are normal.

Exercise 4. If leisure is an inferior good, what is the slope of the supply function of labor?

Answer: Use Slutsky equation to write:

$$\frac{\partial L^S}{\partial w} = \frac{\partial L}{\partial w} + \frac{\partial L}{\partial m}\left(\bar{L} - L\right),$$

where L is leisure, w is wage rate, m is income. Note that the substitution effect is always negative, i.e., $\frac{\partial L^S}{\partial w} < 0$, term $\left(\bar{L} - L\right)$ measures the amount of working hours and it is always positive. Hence, if leisure is a normal good, $\frac{\partial L}{\partial m} > 0$, the sign of the total effect is negative and unambiguous, as the following expression illustrates.

$$\underset{(-)}{\frac{\partial L^S}{\partial w}} = \underset{+!}{\frac{\partial L}{\partial w}} + \underset{(+)}{\frac{\partial L}{\partial m}}\underset{(+)}{\left(\bar{L} - L\right)}$$

In contrast, if leisure is inferior, $\frac{\partial L}{\partial m} < 0$, the total effect, $\frac{\partial L}{\partial w}$, is not necessarily negative. Indeed,

$$\underset{(-)}{\frac{\partial L^S}{\partial w}} = \underset{?}{\frac{\partial L}{\partial w}} + \underset{(-)}{\frac{\partial L}{\partial m}}\underset{(+)}{\left(\bar{L} - L\right)}$$

In order to provide a more general analysis of this case, let us rearrange the equation above, solving for the total effect,

$$\frac{\partial L}{\partial w} = \frac{\partial L^S}{\partial w} - \frac{\partial L}{\partial m}\left(\bar{L} - L\right)$$

Thus, the slope of the labor supply curve depends on whether the total effect is positive or negative, which ultimately depends on whether the (negative) substitution effect dominates the (positive) income effect. Comparing the Substitution and Income effects, and noting that $\frac{\partial(\bar{L} - L)}{\partial w} = -\frac{\partial L}{\partial w}$, then:

1. If $|SE| < |IE|$, then $\frac{\partial L}{\partial w} > 0, and\ \frac{\partial(\bar{L} - L)}{\partial w} < 0$. This implies that the total effect is positive, which implies that the slope of the leisure curve is positive i.e., $\frac{\partial(\bar{L} - L)}{\partial w} < 0$. Therefore the slope of the labor supply curve must be negative, $\frac{\partial L}{\partial w} > 0$.

2. If $|SE| > |IE|$, then $\frac{\partial L}{\partial w} < 0, and\ \frac{\partial(\bar{L} - L)}{\partial w} > 0$. This implies that the total effect is negative, which implies that the leisure curve is negatively sloped i.e., $\frac{\partial(\bar{L} - L)}{\partial w} > 0$. As a consequence, the labor supply curve is positively sloped, $\frac{\partial L}{\partial w} < 0$.