ECON 501 – Micro Theory I

RECITATION #2

MWG 3.D.1. Verify that the Walrasian demand function generated by the Cobb-Douglas utility function satisfies:

(i) Homogeneity of degree zero in *(p,w)*:

- $x(\alpha p, \alpha w) = x(p,w)$ for any p,w and for any scalar $\alpha>0$

(ii) Walras' law:

- $p\cdot x = w$ for all x that belong to $x(p,w)$

(iii) Convexity/Uniqueness: If the preference relation is convex, so that $u(\cdot)$ is quasiconcave, then $x(p,w)$ is a convex set. Moreover, if the preference relation is strictly quasiconvex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p,w)$ consists of a single point.

Solution:

① Assume the Cobb-Douglas utility function to be $U = Ax_1^{\alpha} x_2^{1-\alpha}$.[1] Then, the utility maximization problem (UMP) will be

$$\max_{x_1, x_2} Ax_1^{\alpha} x_2^{1-\alpha}, \text{ subject to } w - p_1 x_1 - p_2 x_2$$

Take FOCs, we can solve for the Walrasian demand function, and get

$$x_1(p,w) = \frac{\alpha w}{p_1}, \text{ and } x_2(p,w) = \frac{(1-\alpha)w}{p_2}$$

② Then we check condition (i). Increasing all prices and wealth by a common factor λ, we obtain

$$x_1(\lambda p, \lambda w) = \frac{\alpha(\lambda w)}{\lambda p_1} = \frac{\alpha w}{p_1} = x_1(p,w),$$

$$x_2(\lambda p, \lambda w) = \frac{(1-\alpha)(\lambda w)}{\lambda p_2} = \frac{(1-\alpha)w}{p_2} = x_2(p,w).$$

[1] All the results in this exercise still hold in the more general case of a Cobb-Douglas utility function $U = Ax_1^{\alpha} x_2^{\beta}$, see MWG 3.D.6 for more details.

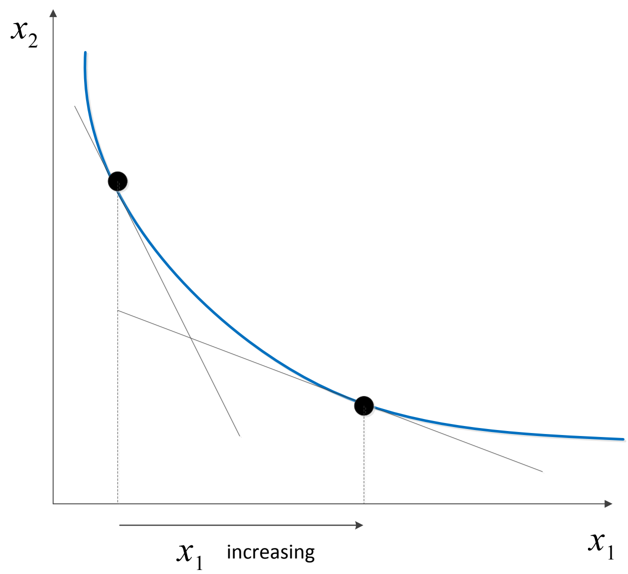
③ We can now check condition (ii),

$$px = p_1 x_1(p,w) + p_2 x_2(p,w) = \frac{p_1 \alpha w}{p_1} + \frac{p_2(1-\alpha)w}{p_2} = w.$$

Thus implying that Walras' law is satisfied.

④ Let us now check condition (iii), i.e., convexity and uniqueness.

Convexity. If a preference relation is convex, the slope of the indifference curve becomes flatter as x_1 increases (rightward moves in the figure).



In order to check that this property holds for the Cobb-Douglas utility function, we need to test whether the MRS (the slope of the indifference curve) is increasing in x_1, e.g., it goes from -5 for low values of x_1 to -1/4 for higher values of x_1. Using the definition of the slope of indifference curves,

$$slope \equiv \frac{dx_2}{dx_1} = -\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} \equiv -MRS_{1,2}$$

we find that, in the case of the Cobb-Douglas utility function, this slope is is

$$-MRS_{1,2} \equiv -\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = -\frac{A\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{A(1-\alpha) x_1^{\alpha} x_2^{-\alpha}} = -\frac{\alpha}{1-\alpha} \cdot \frac{x_2}{x_1}$$

We can now differentiate this expression with respect to x_1 to check if the slope increases in x_1. In particular,

$$\frac{\partial slope}{\partial x_1} = \frac{\partial(-MRS)}{\partial x_1} = \frac{\alpha}{1-\alpha} \cdot \frac{x_2}{x_1^2} > 0$$

Therefore, the slope is indeed increasing in x_1, implying that the preference relation represented with the Cobb-Douglas utility function is convex, i.e., indifference curves are bowed-in towards the origin. In fact, preferences are not only convex, but strictly convex, since an increase in x_1 yields a strict increase of the slope of the indifference curve.

Uniqueness. From the expression of the Walrasian demands found in ①, we know that, for every combination of prices and wealth, (p_1, p_2, w), we obtain a unique quantity demanded for every good, x_1 and x_2. Hence, the Walrasian demand is unique. (Note that his should come as no surprise: when the budget set is convex, i.e., the budget line does not exhibit different slopes, and preferences are strictly convex, indifference curves are tangent to the budget line at a single point, thus producing a unique optimal bundle as Walrasian demand.)

MWG 3.D.2. Verify that the indirect utility function

$$v(p,w)=[\alpha\ln\alpha+(1-\alpha)\ln(1-\alpha)]+\ln w-\alpha\ln p_1-(1-\alpha)\ln p_2$$

satisfies the following properties:

(i) Homogeneous of degree zero in *(p,w)*:

- $v(\alpha p,\alpha w)=v(p,w)$ for any p,w and for any scalar $\alpha>0$

(ii) Strictly increasing in w and nonincreasing in p_l for any l.

(iii) Quasiconvex: the set {(p,w) : v(p,w)≤v} is convex for any v.

(iv) Continuous in p and w.

Solution.

① To check condition (i), we increase all prices and wealth by a common factor λ, which yields

$$
\begin{aligned}
& v(\lambda p, \lambda w) \\
&= \alpha \ln \alpha + (1-\alpha)\ln(1-\alpha) + \ln \lambda w - \alpha \ln \lambda p_1 - (1-\alpha)\ln \lambda p_2 \\
&= \alpha \ln \alpha + (1-\alpha)\ln(1-\alpha) + \ln \lambda + \ln w - \alpha \ln \lambda - \alpha \ln p_1 - (1-\alpha)\ln \lambda - (1-\alpha)\ln p_2 \\
&= \alpha \ln \alpha + (1-\alpha)\ln(1-\alpha) + \ln w - \alpha \ln p_1 - (1-\alpha)\ln p_2 \\
&= v(p, w).
\end{aligned}
$$

② To check condition (ii),

$$
\begin{aligned}
\frac{\partial v(p,w)}{\partial w} &= \frac{1}{w} > 0, \\
\frac{\partial v(p,w)}{\partial p_1} &= -\frac{\alpha}{p_1} < 0, \\
\frac{\partial v(p,w)}{\partial p_2} &= -\frac{1-\alpha}{p_2} < 0.
\end{aligned}
$$

Showing that v(p,w) is strictly increasing in wealth, w, and non-increasing in prices.

③ We can prove that the set $\left\{ p \in \mathbb{R}^L_{++} : v(p,w) \le v \right\}$ is convex.

Recall that to prove Quasiconvexity, we need to show that, for any x,y∈X and $\lambda \in [0,1]$, we have that

$$f(\lambda x + (1-\lambda) y) \le \max\{f(x), f(y)\}$$

while to prove convexity, we need to show that for any x,y∈X and $\lambda \in [0,1]$, we have that

$$f(\lambda x + (1-\lambda) y) \le \alpha f(x) + (1-\alpha) f(y)$$

Define $\Delta = \alpha \ln \alpha + (1-\alpha)\ln(1-\alpha) + \ln w$, and assume

$$\begin{cases} v(p,w) = \Delta - \alpha \ln p_1 - (1-\alpha)\ln p_2 \le \Delta + \bar{V} \\ v(p',w) = \Delta - \alpha \ln p_1' - (1-\alpha)\ln p_2' \le \Delta + \bar{V} \\ (p'',w) = (\lambda p + (1-\lambda)p', w) \end{cases}$$

If we can prove that $v(p'',w) \le \Delta + \bar{V}$, then the set $\{p \in \mathbb{R}^L_{++} : v(p,w) \le v\}$ is convex.

$$v(p'',w) = \Delta - \alpha \ln p_1'' - (1-\alpha)\ln p_2'' = \Delta - \alpha \ln(\lambda p_1 + (1-\lambda)p_1') - (1-\alpha)\ln(\lambda p_2 + (1-\lambda)p_2')$$

$$\begin{cases} \lambda v(p,w) = \lambda\Delta - \alpha\lambda \ln p_1 - (1-\alpha)\lambda \ln p_2 \le \lambda\Delta + \lambda\bar{V} \\ (1-\lambda)v(p',w) = (1-\lambda)\Delta - \alpha(1-\lambda)\ln p_1' - (1-\alpha)(1-\lambda)\ln p_2' \le (1-\lambda)\Delta + (1-\lambda)\bar{V} \end{cases}$$

$$\Rightarrow \lambda v(p,w) + (1-\lambda)v(p',w) = \Delta - \alpha(\lambda \ln p_1 + (1-\lambda)\ln p_1') - (1-\alpha)(\lambda \ln p_2 + (1-\lambda)\ln p_2') \le \Delta + \bar{V}$$

Therefore,

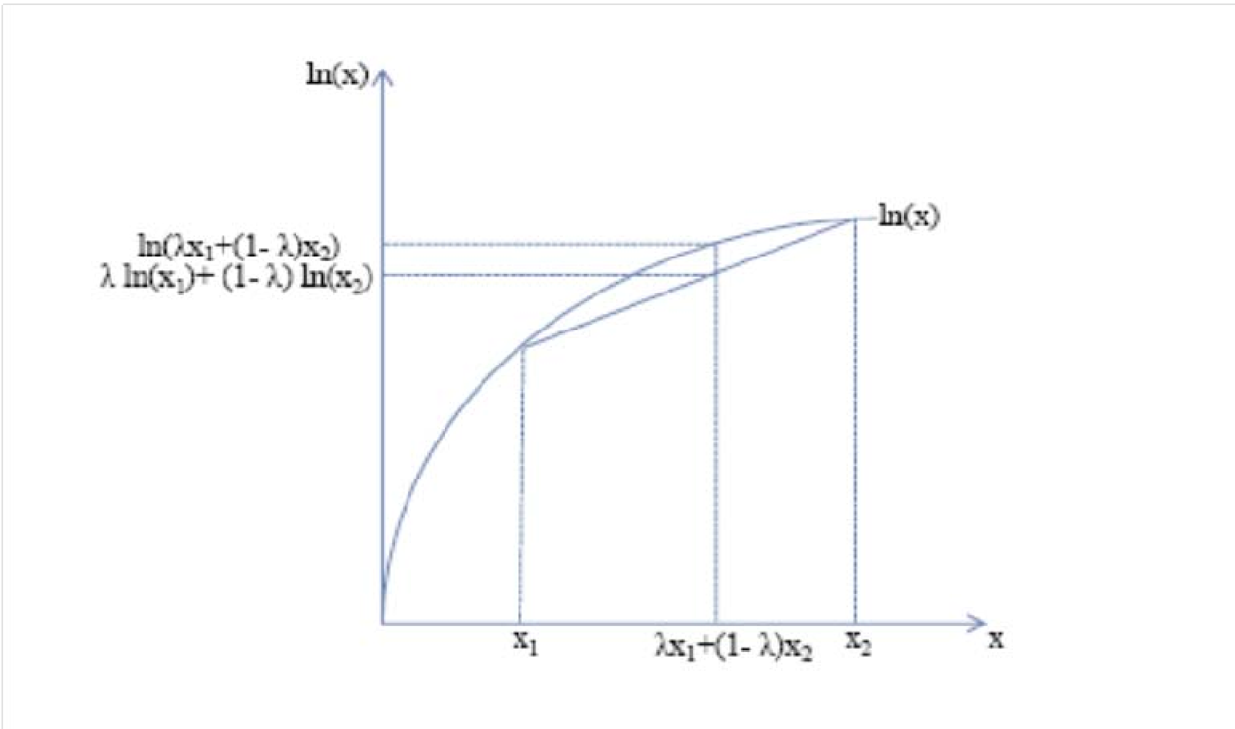
$$v(p'',w) = \Delta - \left[\alpha \ln(\lambda p_1 + (1-\lambda)p_1') + (1-\alpha)\ln(\lambda p_2 + (1-\lambda)p_2')\right]$$

$$\le \Delta - \left[\alpha(\lambda \ln p_1 + (1-\lambda)\ln p_1') + (1-\alpha)(\lambda \ln p_2 + (1-\lambda)\ln p_2')\right]$$

$$\le \Delta + \bar{V}$$

$$\Rightarrow v(p'',w) \le \Delta + \bar{V}$$

(According to the property of concavity of the ln(.), we have $\ln(\lambda x_1 + (1-\lambda)x_2) \ge \lambda \ln x_1 + (1-\lambda)\ln x_2$, as depicted in the figure below)



Therefore, the set $\left\{p \in \mathbb{R}^L_{++} : v\left(p, w\right) \le v\right\}$ is convex.

④ Condition (iv) follows the functional form of $v\left(\cdot\right)$.

MWG 3.D.6. Consider the three good setting in which the consumer has utility function $u(x)=(x_1-b_1)^{\alpha}(x_2-b_2)^{\beta}(x_3-b_3)^{\gamma}$

a) Why can you assume that $\alpha+\beta+\gamma=1$ without loss of generality? Do so for the rest of the problem.

- **Solution.** Define $\tilde{u}\left(x\right) = u\left(x\right)^{\frac{1}{\alpha+\beta+\gamma}} = \left(x_1 - b_1\right)^{\alpha'}\left(x_2 - b_2\right)^{\beta'}\left(x_3 - b_3\right)^{\gamma'}$, with $\alpha' = \frac{\alpha}{\alpha+\beta+\gamma}, \beta' = \frac{\beta}{\alpha+\beta+\gamma}, \gamma' = \frac{\gamma}{\alpha+\beta+\gamma}$. Then $\alpha' + \beta' + \gamma' = 1$ and $\tilde{u}\left(\cdot\right)$ represents the same preferences as $u\left(\cdot\right)$, because the function $u \to u^{\frac{1}{\alpha+\beta+\gamma}}$ is a monotone transformation. Thus we can assume without loss of generality that $\alpha + \beta + \gamma = 1$.

b) Write down the first-order conditions for the UMP, and derive the consumer's Walrasian demand and indirect utility functions. [This system of demands is known as the linear expenditure system, and it is due to Stone (1954)].

- **Solution.** Use another monotone transformation of the given utility function,

$$\bar{u}(x) = \ln u(x) = \alpha \ln(x_1 - b_1) + \beta \ln(x_2 - b_2) + \gamma \ln(x_3 - b_3).$$

$$\ell = \alpha \ln(x_1 - b_1) + \beta \ln(x_2 - b_2) + \gamma \ln(x_3 - b_3) + \lambda(w - p_1 x_1 - p_2 x_2 - p_3 x_3)$$

$$FOCs: \begin{cases} \dfrac{\partial \ell}{\partial x_1} = \dfrac{\alpha}{x_1 - b_1} - \lambda p_1 = 0 \\ \dfrac{\partial \ell}{\partial x_2} = \dfrac{\beta}{x_2 - b_2} - \lambda p_2 = 0 \\ \dfrac{\partial \ell}{\partial x_3} = \dfrac{\gamma}{x_3 - b_3} - \lambda p_3 = 0 \\ \dfrac{\partial \ell}{\partial \lambda} = w - p_1 x_1 - p_2 x_2 - p_3 x_3 = 0 \end{cases} \Rightarrow \begin{cases} p_1 x_1 = \dfrac{\alpha}{\lambda} + p_1 b_1 \\ p_2 x_2 = \dfrac{\beta}{\lambda} + p_2 b_2 \\ p_3 x_3 = \dfrac{\gamma}{\lambda} + p_3 b_3 \end{cases}$$

$$\Rightarrow p_1 x_1 + p_2 x_2 + p_3 x_3 = \frac{\alpha + \beta + \gamma}{\lambda} + p_1 b_1 + p_2 b_2 + p_3 b_3 = w$$

Let $p \cdot b = p_1 b_1 + p_2 b_2 + p_3 b_3$.

$$\Rightarrow \lambda = \frac{\alpha + \beta + \gamma}{w - pb}$$

$$\Rightarrow \begin{cases} x_1 = \dfrac{\alpha}{\lambda p_1} + b_1 = \dfrac{\alpha}{\alpha + \beta + \gamma} \dfrac{(w - pb)}{p_1} + b_1 = \dfrac{\alpha}{p_1}(w - pb) + b_1 \\ x_2 = \dfrac{\beta}{\lambda p_2} + b_2 = \dfrac{\beta}{\alpha + \beta + \gamma} \dfrac{(w - pb)}{p_2} + b_2 = \dfrac{\beta}{p_2}(w - pb) + b_2 \\ x_3 = \dfrac{\gamma}{\lambda p_3} + b_3 = \dfrac{\gamma}{\alpha + \beta + \gamma} \dfrac{(w - pb)}{p_3} + b_3 = \dfrac{\gamma}{p_3}(w - pb) + b_3 \end{cases}$$

$$(\alpha + \beta + \gamma = 1)$$

Get demand function $x(p, w) = (b_1, b_2, b_3) + (w - p \cdot b)\left(\dfrac{\alpha}{p_1}, \dfrac{\beta}{p_2}, \dfrac{\gamma}{p_3}\right)$

Plug into $u(x) = (x_1 - b_1)^{\alpha}(x_2 - b_2)^{\beta}(x_3 - b_3)^{\gamma}$, then we obtain the indirect utility function

$$v(p,w)=(w-p\cdot b)\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{\gamma}{p_3}\right)^{\gamma}.$$

c) Verify that these demand functions satisfy the properties listed in Propositions 3.D.2 for the Walrasian demand function, and in Proposition 3.D.3 for the indirect utility function.

① To check the homogeneity of the demand function,

$$x(\lambda p,\lambda w)=(b_1,b_2,b_3)+(\lambda w-\lambda p\cdot b)\left(\frac{\alpha}{\lambda p_1},\frac{\beta}{\lambda p_2},\frac{\gamma}{\lambda p_3}\right)$$

$$=(b_1,b_2,b_3)+(w-p\cdot b)\left(\frac{\alpha}{p_1},\frac{\beta}{p_2},\frac{\gamma}{p_3}\right)=x(p,w).$$

To check Walras law,

$$\begin{aligned}p\cdot x(p,w)&=p_1\cdot x_1(p,w)+p_2\cdot x_2(p,w)+p_3\cdot x_3(p,w)\\&=p_1b_1+p_2b_2+p_3b_3+(w-p\cdot b)\left(\frac{p_1\alpha}{p_1}+\frac{p_2\beta}{p_2}+\frac{p_3\gamma}{p_3}\right)\\&=p\cdot b+(w-p\cdot b)(\alpha+\beta+\gamma)=w.\end{aligned}$$

The uniqueness is obvious.

② To check the homogeneity of the indirect utility function,

$$v(\lambda p,\lambda w)=(\lambda w-\lambda p\cdot b)\left(\frac{\alpha}{\lambda p_1}\right)^{\alpha}\left(\frac{\beta}{\lambda p_2}\right)^{\beta}\left(\frac{\gamma}{\lambda p_3}\right)^{\gamma}$$

$$=\lambda^{1-(\alpha+\beta+\gamma)}(w-p\cdot b)\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{\gamma}{p_3}\right)^{\gamma}$$

$$=(w-p\cdot b)\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{\gamma}{p_3}\right)^{\gamma}=v(p,w).$$

To check the monotonicity,

$$\frac{\partial v(p,w)}{\partial w} = \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{\gamma}{p_3}\right)^{\gamma} > 0,$$

$$\frac{\partial v(p,w)}{\partial p_1} = v(p,w)\cdot\left(-\frac{\alpha}{p_1}\right) < 0,$$

$$\frac{\partial v(p,w)}{\partial p_2} = v(p,w)\cdot\left(-\frac{\beta}{p_2}\right) < 0,$$

$$\frac{\partial v(p,w)}{\partial p_3} = v(p,w)\cdot\left(-\frac{\gamma}{p_3}\right) < 0.$$

The continuity follows directly from the given functional form.

In order to prove the quasiconvexity, it is sufficient to prove that, for any $v \in \mathbb{R}$ and $w > 0$, the set $\left\{p \in \mathbb{R}^3 : v(p,w) \le v\right\}$ is convex. Consider the monotone transformation

$$\ln v(p,w) = \alpha \ln \alpha + \beta \ln \beta + \gamma \ln \gamma + \ln(w - p \cdot b) - \alpha \ln p_1 - \beta \ln p_2 - \gamma \ln p_3.$$

Since the logarithmic function is concave, the set

$$\left\{p \in \mathbb{R}^3 : \ln(w - p \cdot b) - \alpha \ln p_1 - \beta \ln p_2 - \gamma \ln p_3 \le v\right\}$$

is convex for every $v \in \mathbb{R}$. Since the other terms, $\alpha \ln \alpha + \beta \ln \beta + \gamma \ln \gamma$, do not depend on p, this implies that the set $\left\{p \in \mathbb{R}^3 : \ln v(p,w) \le v\right\}$ is convex. Hence so is $\left\{p \in \mathbb{R}^3 : v(p,w) \le v\right\}$.

(Or follow the same step as in MWG 3.D.2 ③.)

MWG 3.D.7. There are two commodities. We are given two budget sets B_{p^0,w^0} and B_{p^1,w^1} described, respectively, by

$$p^0=(1,1) \text{ and } w^0=8$$

$$p^1=(1,4) \text{ and } w^1=26$$

The observed choice at (p^0, w^0) is x^0=(4,4), and at (p^1, w^1) the consumer's choice satisfies $p \cdot x^1 = w$.

a) Determine the region of permissible choices for x^1, if the choices x^0 and x^1 are consistent with maximization of preferences.

- Before applying the definition of WARP to the following figure, we should clarify WARP in this context: The Walrasian demand satisfies WARP if, for any two bundles x^0 and x^1, both of them being affordable under the new budget line $B_{p1,w1}$, the new bundle x^1 is unaffordable under the old budget line, $B_{p0,w0}$. (Note that, relative to the definition of WARP we discussed in class, here we are just switching the new and old budget line and, as a consequence, the bundle that needs to be unaffordable.)

- Since x^0 is affordable under $B_{p1,w1}$, $p^1 \cdot x^0 < w^1$ and $x^1 \neq x^0$, (indeed, bundle (4,4) lies strictly below budget line $B_{p1,w1}$) the premise of WARP holds, and we can move to the second step. For the second step of WARP to hold, we need that bundle x^1 must be unaffordable, i.e., $p^0 \cdot x^1 > w^0$. Thus, x^1 has to be on the bold thick line in the following figure.

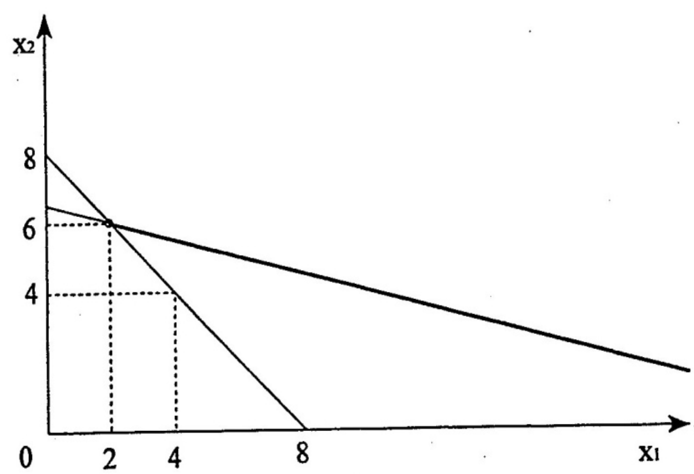


Figure 3.D.7(a)

In the following four questions, we assume the given preference can be a differentiable utility function $u(\cdot)$.

b) Determine the region of permissible choices for x^1, if the choices x^0 and x^1 are consistent with maximization of preferences that are quasilinear with respect to the *first* good.

- If the preference is quasilinear with respect to the first good, then we can take a utility function $u(\cdot)$ so that $\frac{\partial u(x)}{\partial x_1} = 1$ for every x (Exercise 3.C.5(b)). Hence the first-order condition implies $\frac{\partial u(x^t)}{\partial x_2^t} = \frac{p_2^t}{p_1^t}$ for each $t = 0,1$. Since $\frac{p_2^0}{p_1^0} < \frac{p_2^1}{p_1^1}$ and $u(\cdot)$ is concave, $x_2^0 > x_2^1$. Thus x^1 has to be on the bold line in the following figure.

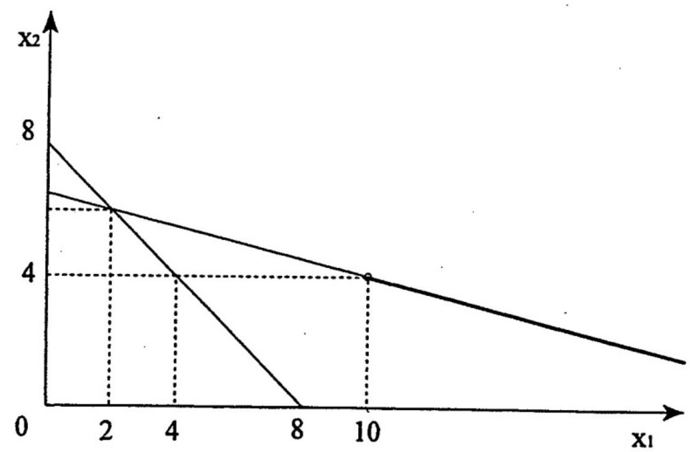


Figure 3.D.7(b)

c) Determine the region of permissible choices for x^1, if the choices x^0 and x^1 are consistent with maximization of preferences that are quasilinear with respect to the *second* good.

- If the preference is quasilinear with respect to the second good, then we can take a utility function $u(\cdot)$ so that $\frac{\partial u(x)}{\partial x_2} = 1$ for every x (Exercise 3.C.5(b)). Hence the first-order condition implies $\frac{\partial u(x^t)}{\partial x_1^t} = \frac{p_1^t}{p_2^t}$ for each $t = 1,0$. Since $\frac{p_1^0}{p_2^0} > \frac{p_1^1}{p_2^1}$ and $u(\cdot)$ is concave, we must have $x_1^0 < x_1^1$. Thus x^1 has to be on the bold line in the following figure.

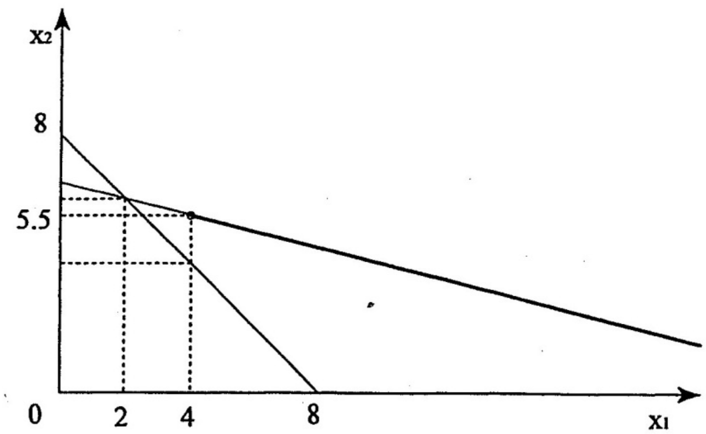


Figure 3.D.7(c)

d) Determine the region of permissible choices for x^1, if the choices x^0 and x^1 are consistent with maximization of preferences <u>for which both goods are normal</u>.

- Since $p^1 \cdot x^0 < w^1$ and the relative price of good 1 decreased, x_1^t has to increase if good 1 is normal. If good 2 is normal, then the wealth effect (positive) and the substitution effect (negative) go in opposite direction which gives us no additional information about x_2. Thus x^1 has to be on the bold line in the following figure.

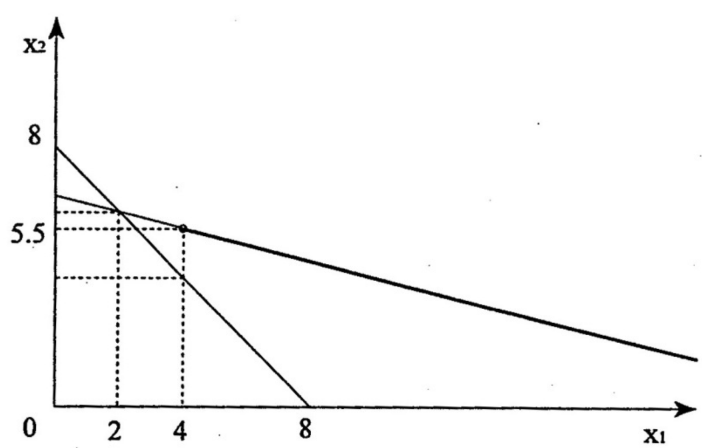


Figure 3.D.7(d)

e) Determine the region of permissible choices for x^1, if the choices x^0 and x^1 are consistent with maximization of preferences when preferences are homothetic.

- If the preference is homothetic, the marginal rates of substitution at all vectors on a ray are the same, and they become less steep as the ray becomes flatter. By the first-order conditions and $\frac{p_1^0}{p_2^0} > \frac{p_1^1}{p_2^1}$, x^1 has to be on the right side of the ray that goes through x^0.

  Thus x^1 has to be on the bold line in the following figure.

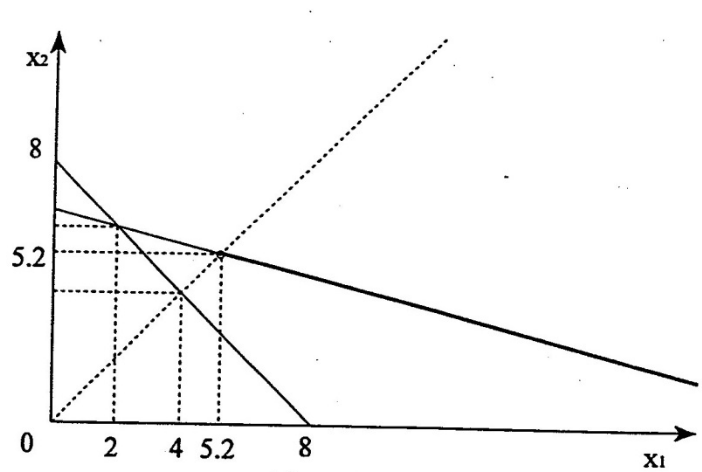


Figure 3.D.7(e)