Micro Theory I
Recitation #1 - Preferences and Choice[1]

1. **MWG 1.B.2.** Prove property (ii) of Proposition 1.B.1: If a preference relation $\succsim$ is rational (i.e., satisfied completeness and transitivity) then:

   (a) $\sim$ is reflexive ($x \sim x$ for all x), transitive (if $x \sim y$ and $y \sim z$, then $x \sim z$), and symmetric (if $x \sim y$, then $y \sim x$).

   - *Reflexivity:* Since $x \succsim x$ for every $x \in X$, $x \sim x$ for every $x \in X$ as well. Thus $\sim$ is reflexive.
   - *Transivity:* Suppose that $x \sim y$ and $y \sim z$. Then, on one hand, $x \succsim y$ and $y \succsim x$; and, on the other hand, $y \succsim z$ and $z \succsim y$. By transitivity of the preference relation $\succsim$, this implies that $x \succsim z$ and $z \succsim x$, ultimately entailing $x \sim z$. Hence $\sim$ is transitive.
   - *Symmetry:* Suppose that $x \sim y$. Then, it must be that $x \succsim y$ and $y \succsim x$. Thus, we must also have that $y \succsim x$ and $x \succsim y$, which entails that the individual is indifferent between y and x, $y \sim x$. Therefore, $\sim$ is symmetric.

   b. if $x \succ y \succsim z$, then $x \succ z$.

   - Let's prove this result by contradiction. That is, we assume that the premise holds (the initial assumption $x \succ y \succsim z$ is satisfied), but the conclusion of the statement, $x \succ z$, does not hold, i.e., $z \succsim x$. In short, we seek to show that

     $$\text{if } x \succ y \succsim z \text{ then } z \succsim x$$

     leads to a contradiction. To identify the contradiction, first note that if $y \succsim z$ and $z \succsim x$, by transitify, $y \succsim x$. (We can use transitivity because the preference relation is rational.) But our last result ($y \succsim x$) contradicts the first part of our initial assumption $x \succ y \succsim z$, i.e., we cannot have $y \succsim x$ and $x \succ y$ simultaneously satisfied. Hence, we have reached a contradiction. Therefore, if a rational preference relation satisfies $x \succ y \succsim z$, it must be that $x \succ z$ holds too.

2. **MWG 1.C.1.** Consider the choice structure $(\mathcal{B}, C(\cdot))$ with $B = (\{x, y\}, \{x, y, z\})$ and $C(\{x, y\}) = \{x\}$. Show that if $(\mathcal{B}, C(\cdot))$ satisfies the weak axiom, then we must have $C(\{x, y, z\}) = \{x\}, = \{z\}, or = \{x, z\}$.

   - Recall that the choice structure $(\mathcal{B}, C(\cdot))$ satisfies the WARP if,
     - for some budget set $B \in \mathcal{B}$ with $x, y \in B$ we have that element x is chosen, $x \in C(B)$, then
     - for any other budget set $B' \in \mathcal{B}$ where alternatives x and y are also available, $x, y \in B'$, and where alternative y is chosen, $y \in C(B')$, then we must have that alternative x is chosen as well, $x \in C(B')$.

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- Then, if $y \in C(\{x,y,z\})$, then the WARP would imply that $y \in C(\{x,y\})$. But contradicts the equality $C(\{x,y\}) = \{x\}$. Hence $y \notin C(\{x,y,z\})$. Thus

$$\begin{aligned} \text{either } C(\{x,y,z\}) &= \{x\}, \text{ or} \\ C(\{x,y,z\}) &= \{z\}, \text{ or} \\ C(\{x,y,z\}) &= \{x,z\}, \text{ but} \\ C(\{x,y,z\}) &\neq \{y\} \end{aligned}$$

3. **MWG 2.D.3(b).** Consider an extension of the Walrasian budget set to an arbitrary consumption set X such that

$$B_{p,w} = \{x \in X : p \cdot x \leq w\}.$$

Show that if X is a convex set, then $B_{p,w}$ is convex as well.

- Let $x \in B_{p,w}$ and $x' \in B_{p,w}$. Now consider the linear combination of these two bundles $x'' = \lambda x + (1-\lambda)x'$ where $\lambda \in [0,1]$. Since X is convex, $x'' \in X$. Moreover,

$$p \cdot x'' = \lambda(p \cdot x) + (1-\lambda)(p \cdot x') \leq \lambda w + (1-\lambda) w = w$$

Thus $x'' \in B_{p,w}$.

4. **MWG 2.D.4** Show that the budget set in Figure 2.D.4 is not convex.

- It follows from a direct calculation that consumption level M can be attained by working $8 + \frac{M-8s}{s'}$ hours. It follows from the definition that $A = (24, 0)$, the horizontal intercept, and $B = \left(16 - \frac{M-8s}{s'}, M\right)$ are both on the budget line (the latter point is the kink where the worker obtains a wealth of M, as we explain below).
  - *Obtaining the kink.* First, note that for the first 8 hours of work (reducing his leisure from 24 to 16 hours) the worker obtains $8s$ since his salary is s per hour. Second, since the distance between M and $8s$ in the vertical axis is $M - 8s$, the distance in the horizontal axis becomes $\frac{M-8s}{s'}$. Therefore, the amount of leisure at the upper kink (corresponding to a consumption of M) is $16 - \frac{M-8s}{s'}$.
- We next seek to show that the convex combination of these two consumption vectors yields a point that lies above $8s$, so the budget set is *not* convex. (Recall that to show that convexity is violated we only need to find one point on the convex combination between A and B that doesn't belong to the budget set.) For instance, let's identify the weight that yields a leisure of 16 hours, as follows

$$24\alpha + (1-\alpha)\left(16 - \frac{M-8s}{s'}\right) = 16$$

which, solving for α, yields

$$\alpha = \frac{\frac{M-8s}{s'}}{8+\frac{M-8s}{s'}}$$

Hence, when $\alpha = \frac{\frac{M-8s}{s'}}{8+\frac{M-8s}{s'}}$, the point connecting bundle $(24, 0)$ and $\left(16 - \frac{M-8s}{s'}, M\right)$ lies at exactly 16 hours of leisure in the horizontal axis. We next find the corresponding consumption level (on the vertical axis), which is

$$\begin{aligned} 0\alpha + M(1-\alpha) &= 0\frac{\frac{M-8s}{s'}}{8+\frac{M-8s}{s'}} + M\left(1 - \frac{\frac{M-8s}{s'}}{8+\frac{M-8s}{s'}}\right) \\ &= M\frac{8}{8+\frac{M-8s}{s'}} \end{aligned}$$

- Finally, we only need to show that the height of this point is larger than $8s$. In order to make this comparison, note that $8s$ can be alternatively expressed as

$$8s = M\frac{8}{\frac{M}{s}} = M\frac{8}{8+\frac{M-8s}{s}}$$

where in the first equality we multiply by $\frac{M}{M}$ and by $\frac{1/s}{1/s}$; and in the second equality, we add and subtract $8s$ in the denominator. We can now compare the consumption level of 16 hours of leisure, $M\frac{8}{8+\frac{M-8s}{s'}}$, against the point on the budget line, $8s$, and obtain

$$M\frac{8}{8+\frac{M-8s}{s'}} > M\frac{8}{8+\frac{M-8s}{s}} = 8s$$

since $s' > s$ by definition. Therefore, the budget set is not convex.

5. **MWG 3.B.1** Show the following:

   (a) If $\succsim$ is strongly monotone, then it is monotone.

   (b) If $\succsim$ is monotone, then it is locally nonsatiated.

- Before we start, let us recall the definition of monotonicity, strong monotonicity and LNS.

  – A preference relation satisfies *monotonicity* if, for all $x, y \in X$, $x_k \geq y_k$ for all k implies $x \succsim y$, and if $x_k > y_k$ for all k (i.e., $x >> y$) then $x \succ y$.

  – A preference relation satisfies *strong monotonicity* if, for all $x, y \in X$, $x_k \geq y_k$ for all k implies $x \succ y$. [Intuitively, increasing one of the components of bundle x yields a new bundle y that is strictly preferred to bundle x]

  – A preference relation satisfies *LNS* if, for all $x \in X$ and $\epsilon > 0$, there exists another bundle $y \in X$ such that $\|y - x\| \leq \epsilon$ and $y \succ x$.

- Answer: *(a)* Assume that $\succsim$ is strongly monotone and $x >> y$, i.e., bundle x is higher than bundle y in every component. Then $x \geq y$ and $x \neq y$. Hence $x \succ y$. Thus $\succsim$ is monotone.

- Answer *(b)*. Assume that $\succsim$ is monotone, $x \in X$, and $\epsilon > 0$. Let $e = (1, \ldots, 1) \in \mathbb{R}^L$ and let $y = x + \frac{\epsilon}{\sqrt{L}}e$. Then

$$\|y - x\| = \sqrt{(y_1 - x_1)^2 + ... + (y_L - x_L)^2} = \sqrt{\left(\frac{\epsilon}{\sqrt{L}}\right)^2 L} = \sqrt{\frac{\epsilon^2}{L}L} = \epsilon$$

i.e., $\|y - x\| \leq \epsilon$ holds, and $y \succ x$. Thus $\succsim$ is locally nonsatiated.

6. **MWG 3.B.3** Draw a convex preference relation that is locally nonsatiated but is not monotone.

   - Recall that a monotone preference relation satisfies LNS, but the converse is not necessarily true.
   - Following is an example of a convex, locally nonsatiated preference relation that is not monotone in $\mathbb{R}^2_+$. For example, $x >> y$ but $y \succ x$. Indeed, this preference relation is not monotone since, despite $x >> y$ (i.e., bundle x contains more of all goods than bundle y), and yet $y \succsim x$, since it lies on an indifference curve associated to higher utility levels than x.