EconS 424 - Strategy and Game Theory

Handout on Rationalizability and IDSDS[1]

1 Introduction

In this handout, we will discuss an extension of best response functions: Rationalizability.

Best response: As we have already covered, we know that we can define s_i to be a best response of player i to a profile of strategies s_{-i} played by his opponents, $s_{-i} \equiv (s_1, s_2, ..., s_I)$, if it yields a larger utility than any other of his available strategies $s_i' \in S_i$, that is

$$v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \quad \text{for all } s_i' \in S_i$$

Rationalizability: A rationalizable strategy is a strategy that is a best response for a player, given some beliefs about other players's behavior. We can define a belief as some probability, p, that a player plays a particular strategy. Formally, a strategy is rationalizable if

$$s_i \in \overbrace{BR_i(p)}^{\substack{\text{Best Response function} \\ \text{conditioned on beliefs}}}$$

which says that strategy $s_i \in S_i$ is a best response function given beliefs p. We will apply rationalizability in order to eliminate strategies that are never a best response (i.e., that are not rationalizable as best responses for **any** beliefs that player i might sustain about his opponent's behavior).

Strictly dominant strategies: A strictly dominant strategy $s_i \in S_i$ for player i yields a strictly larger payoff than all of his available strategies $s_i' \in S_i$, and does so for all possible strategies played by his opponents. Formally,

$$v_i(s_i, s_{-i}) > v_i(s_i', s_{-i}) \quad \text{for all } s_i' \in S_i, \text{ and for all } s_{-i} \in S_{-i},\ s_i' \neq s_i$$

When this happens, we can assign a zero probability to our beliefs that this strategy will be a best response for player i, and it will therefore not be rationalizable. Effectively, we can delete a strictly dominated strategy from a normal form game, as it will never be played by a rational player. Recall that eliminating all strictly dominated strategies for all players is referred to as IDSDS.

In the next examples, we explore settings in which applying IDSDS provides the same set of equilibrium predictions as applying rationalizability (as in the first example), and in which settings

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the two equilibrium concepts differ (i.e., and analyze which one offers more precise equilibrium predictions); as illustrated in the second example.

2 Examples

First example. We will first find equilibrium predictions applying IDSDS, and then repeat the process, but this time applying rationalizability (i.e., eliminating those strategies that are not best responses).

		Player 2 L	Player 2 C	Player 2 R
	U	2, 0	2, 0	3, -1
Player 1	M	3, 1	0, 0	2, -1
	D	0, 0	3, 1	2, -1

IDSDS. As can be seen in the figure, strategy R is never chosen by player 2: it yields a payoff of -1 regardless of the strategy (row) selected by player 1, while selecting either L or C yields payoffs larger than -1. Hence, no matter what strategy player 1 plays, picking either strategy L or C strictly dominates strategy R. Hence, we say that strategy R is *strictly dominated* and can be thus deleted from this game (since a rational player 2 would never play it). Our new reduced form game is shown in the next figure.

		Player 2 L	Player 2 C
	U	2, 0	2, 0
Player 1	M	3, 1	0, 0
	D	0, 0	3, 1

We now examine whether player 1 has strictly dominated strategies we could delete from the reduced-form matrix. Unfortunately, we cannot find any strictly dominated strategies for player 1. Hence, the application of IDSDS would leave us with the above reduced-form matrix, with six different strategy profiles, as the most precise equilibrium prediction we can provide, i.e.,

$$\{(U, L), (U, C), (M, L), (M, C), (D, L), (D, C)\}$$

Rationalizability. What about rationalizability? Does it provide any more precise equilibrium predictions? We can start by investigating whether there are any conditions in which picking strategy U is the best response for player 1, that is, we need to determine if strategy U is rationalizable. Recall that a rationalizable strategy is a strategy that is the best response for a player, given some beliefs about the other players. In our game, let's define p as the probability (belief) that player 2 is going to play strategy L. This would imply that the probability that player 2 plays strategy C will be $1-p$ (since probabilities add up to 1). Putting these value into our game, we obtain in the next figure

		Player 2	
		p L	$1 - p$ C
	U	2, 0	2, 0
Player 1	M	3, 1	0, 0
	D	0, 0	3, 1

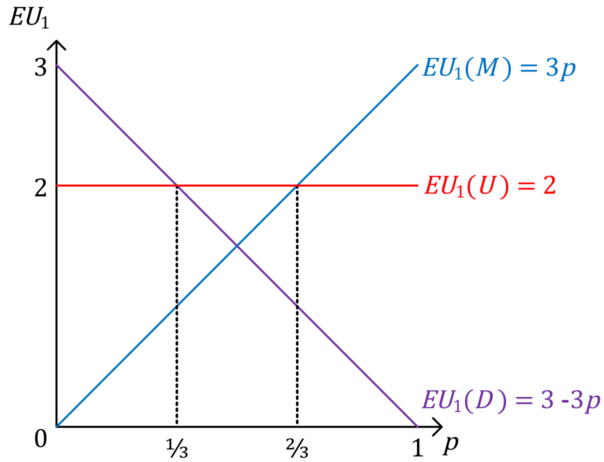
If player 2 is playing each of his strategies, L and C, with a probabilities p and $1-p$, respectively, then player 1 can calculate his expected utility for each of his own strategies. This is done by multiplying each outcome for player 1's strategy by the probability it occurs, then adding it up. Player 1's expected utilities are then

$$\overbrace{EU_1(U)}^{\substack{\text{Expected utility}\\ \text{of player 1 when}\\ \text{he plays } U}} = \overbrace{2}^{\substack{\text{Player 2}\\ \text{plays } L}} *p + \overbrace{2}^{\substack{\text{Player 2}\\ \text{plays } C}} *(1-p) = 2$$

$$EU_1(M) = 3*p + 0*(1-p) = 3p$$

$$EU_1(D) = 0*p + 3*(1-p) = 3-3p$$

We can graph all of these expected utilities, as shown in the next figure.



For strategy U to be rationalizable, we must be able to find at least one value for p such that the expected utility of playing strategy U is at least as good as the expected utilities of playing both strategies M and D. Formally, we need $EU_1(U) \geq EU_1(M)$ and $EU_1(U) \geq EU_1(D)$ at the same time for at least one value of p. Solving for these inequalities, we obtain

$$\begin{aligned} EU_1(U) &\geq EU_1(M) \\ 2 &\geq 3p \Longleftrightarrow p \leq \frac{2}{3} \\ EU_1(U) &\geq EU_1(D) \\ 2 &\geq 3 - 3p \Longleftrightarrow p \geq \frac{1}{3} \end{aligned}$$

Hence, any belief p that satisfies $\frac{1}{3} \leq p \leq \frac{2}{3}$ will make player 1's best response to choose strategy U (We can also see this on our graph where the line depicting $EU_1(U)$ is above both of the other lines). For example, if player 1 believed player 2 would play strategy L 50% of the time ($p = 0.5$), then his expected utilities would be

$$\begin{aligned} EU_1(U) &= 2 \\ EU_1(M) &= 3 * (0.5) = 1.5 \\ EU_1(D) &= 3 - 3 * (0.5) = 1.5 \end{aligned}$$

which clearly shows that choosing strategy U is player 1's best response when $p = 0.5$. Hence, since there exist beliefs for player 1 where strategy U is his best response, we say that strategy U is rationalizable. We can also represent player 1's best response function as a function of his beliefs:

$$BR_1(p) = \begin{cases} U \text{ if } \frac{1}{3} \leq p \leq \frac{2}{3} \\ M \text{ if } p \geq \frac{2}{3} \\ D \text{ if } p \leq \frac{2}{3} \end{cases}$$

This means that all strategies for player 1 are rationalizable, and the set of rationalizable strategies is the same in this problem as given by IDSDS, meaning our equilibrium prediction cannot be refined any further.

Second example. Let's look at one more example. Consider the game depicted in the following figure.

		Player 2 p L	Player 2 $1 - p$ C
Player 1	U	2, 0	2, 0
	M	5, 1	0, 0
	D	0, 0	5, 1

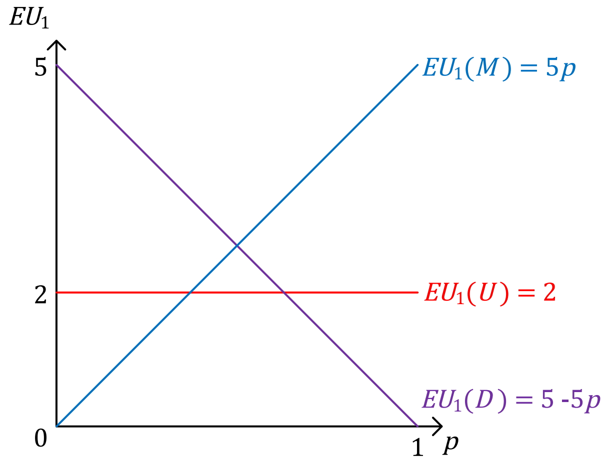
IDSDS. First, note that the application of IDSDS doesn't have a bite, yielding the six possible outcomes

$$\{(U, L), (U, C), (M, L), (M, C), (D, L), (D, C)\}$$

Rationalizability. However, the application of rationalizability has more predictive power (more 'bite') than IDSDS. Let us determine if strategy U is rationalizable. This time, however, player 1's payoff from strategies M and D is much higher. As before, we assign probability p to player 2 playing strategy L and calculate player 1's expected utilities.

$$\begin{aligned} EU_1(U) &= 2 * p + 2 * (1 - p) = 2 \\ EU_1(M) &= 5 * p + 0 * (1 - p) = 5p \\ EU_1(D) &= 0 * p + 5 * (1 - p) = 5 - 5p \end{aligned}$$

Graphing these values,



Like before, we need to find values p for which $EU_1(U) \geq EU_1(M)$ and $EU_1(U) \geq EU_1(D)$. Solving for these inequalities, yields

$$\begin{aligned} EU_1(U) &\geq EU_1(M) \\ 2 &\geq 5p \iff p \leq \frac{2}{5} \\ EU_1(U) &\geq EU_1(D) \\ 2 &\geq 5 - 5p \iff p \geq \frac{3}{5} \end{aligned}$$

In this case, there does not exist a value for p that is both less than $\frac{2}{5}$ and greater than $\frac{3}{5}$ at the same time (As we can see on our graph, there is no point in which the line for $EU_1(U)$ is above the other two lines). Hence, there are no beliefs for player 1 in which strategy U is the best response and thus strategy U is not rationalizable. In this situation, player 1 will never choose strategy U under any circumstances and we can delete it from our game, providing the following reduced form matrix as shown in the next figure.

		Player 2	
		L	C
Player 1	M	5, 1	0, 0
	D	0, 0	5, 1

At this point, we cannot delete any other strategies for player 1 or 2 (as never being best responses). In particular, for player 1, M is a best response to L, while D is a best response to C. Similarly, for player 2 L is a best response to M, and C is a best response to D. While we cannot eliminate any further strategies, in this case, rationalizability gives a more precise equilibrium prediction than IDSDS, namely

$$\{(M, L), (M, C), (D, L), (D, C)\}$$