EconS 503 – Advanced Microeconomics II

Moral Hazard – *Continuous effort levels*

Let us consider again a setting in which a principal hires an agent. In particular, the utility function of the agent is

$$u(w,e) = E[w] - \frac{1}{2}\rho Var[w] - c(e)$$

where ρ measures the Arrow-Pratt coefficient of absolute risk aversion, and where $e \in [0,1]$ measures his effort, where $c = \frac{1}{2}e^2$. (Recall that CARA utility functions, such as $u(w) = -\exp^{-\rho w}$, exhibit a constant Arrow-Pratt coefficient of risk aversion $r_A(w) \equiv -\frac{u''(w)}{u'(w)} = \rho$.)

The outcome of the project , x, is stochastic and given by

$$x = f(e,\varepsilon) = e + \varepsilon, \quad \text{where } \varepsilon \sim N\,(0,\sigma^2).$$

The agent's reservation utility is $\bar{u} = 0$

The principal offers a linear contract to the agent

$$w(x) = a + bx$$

where $a, b > 0$, and a can be interpreted as a fixed payment whereas b provides the agent with incentives to achieve good outcomes (higher *x*).

Let us next find the optimal contract that the principal offers the agent.

- First, note that expected profits are

$$E[\pi] = E[x - w] = E[x] - E[w]$$

$$= e - (a + b\underbrace{E[x]}_{e}) = e - a - be$$

$$= (1-b)e - a$$

- Second, the expected utility of the agent when he exits effort level e is

$$E[u(w,e)] = \underbrace{E[w]}_{a+be} - \frac{1}{2}\rho \underbrace{Var[w]}_{Van[x] = b^2\sigma^2} - \underbrace{c(e)}_{\frac{1}{2}e^2}$$

$$= a + be - \frac{1}{2}\rho b^2\sigma^2 - \frac{1}{2}e^2$$

Hence, the principal's problem is to choose the fixed payment , a, and the bonus , b, to solve

$\max_{\{e,a,b\}} \quad (1-b)e - a$

subject to $\quad a + be - \frac{1}{2}\rho b^2\sigma^2 - \frac{1}{2}e^2 \geq 0 \quad$ PC

$\frac{\partial E[u(w,e)]}{\partial e} = b - e = o \rightarrow e = b \quad$ IC

The IC in exercises with a continuum of effort levels (or any other continuous choice variable for the agent) consists of solving his UMP once he has received the contract from the principal. That's why we differentiate the agent's expected utility with respect to *e* so the agent chooses his utility-maximizing effort level.

Remark: Using the FOC of the agent's UMP as the IC of the principal problem (as we did in the above problem) is commonly known as the "first order approach" since we use first-order conditions as the IC. Note that, while the FOCs are necessary conditions for a maximum of the agent's UMP, they are not sufficient conditions. However, in most applications, the FOCs will be sufficient. For examples of principal-agent problems in which the first-order approach is not valid (because FOCs produce a minimum, rather than a maximum) see Bolton and Dewatripont (Chapter 4, subsection 4.4.2, scanned pages at the end of this handout).

Plugging the result that we found from the IC, $e = b$, into the above program yields a problem with only two choice variables and only one constraint:

$\max_{\{a,b\}} \quad (1-b)b - a$

subject to $\quad a + b^2 - \frac{1}{2}\rho b^2\sigma^2 - \frac{1}{2}b^2 \geq 0$

With associated Lagragean

$$\mathcal{L} = b - a + \lambda[a + b^2 - \tfrac{1}{2}\rho b^2\sigma^2 - \tfrac{1}{2}b^2]$$

Kuhn-Tucker conditions are:

$$\frac{\partial \mathcal{L}}{\partial a} = -1 + \lambda = 0 \rightarrow \lambda = 1 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial b} = 1 - 2b + \lambda[2b - \rho b\sigma^2 - b] = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} = a + b^2 - \frac{1}{2}\rho b^2\sigma^2 - \frac{1}{2}b^2 = 0 \qquad (3)$$

From (1), $\lambda = 1$. Plugging this result into (2) yields

$$1 - 2b + 2b - \rho b\sigma^2 - b = 0 \rightarrow 1 = b(1 + \rho\sigma^2)$$

and solving for the incentive b we obtain

$$b = \frac{1}{1+\rho\sigma^2}$$

which is decreasing in the agent's risk aversion parameter, ρ, and in the variance of outcomes, σ^2.

Let us now find the remaining unknown, i.e., the fixed payment a.

We can find it by using the PC constraint, which binds since $\lambda = 1$.

In particular,

$$a + \underbrace{\left(\frac{1}{1+\rho\sigma^2}\right)}_{b}^2 - \frac{1}{2}\rho\underbrace{\left(\frac{1}{1+\rho\sigma^2}\right)}_{b}^2\sigma^2 - \frac{1}{2}\left(\frac{1}{1+\rho\sigma^2}\right)^2 = 0$$

Rearanging

$$a + \left(\frac{1}{1+\rho\sigma^2}\right)^2\left[\underbrace{1 - \frac{1}{2}\rho\sigma^2 - \frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}\rho\sigma^2 = \frac{1}{2}(1-\rho\sigma^2)}\right] = 0$$

And solving for the fixed payment a, yields

$$a = -\left(\frac{1}{1+\rho\sigma^2}\right)^2\frac{1}{2}(1-\rho\sigma^2)$$

Extreme cases

If $\sigma^2 = 0$ so effort is deterministic, i.e., $x = f(e) = e$, then

$$b = \frac{1}{1+\rho\cdot 0} = 1$$

$$a = -\left(\frac{1}{1+\rho\cdot 0}\right)^2\frac{1}{2}(1-\rho\cdot 0) = -\frac{1}{2}$$

If $\sigma^2 = 1$ so effort is a very imprecise predictor of outcomes,

$$b = \frac{1}{1+\rho}$$

$$a = 1\left(\frac{1}{1+\rho}\right)^2 \frac{1}{2}(1-\rho)$$

More generally, there is a negative relationship between a and b as σ^2 increases:

- when σ^2 is low the fixed payment *a* is low (or even negative) while the bonus *b* is high.
- when σ^2 is high the fixed payment *a* is high (although it can still be negative) wheras the bonus *b* is low.

$$\frac{1}{u'[w(q)]}=\lambda+\mu\left[1-\frac{f(q\mid a_L)}{f(q\mid a_H)}\right]$$

We know that $u'(\cdot)>0$, so that

$$\frac{dw}{dq}>0\Leftrightarrow\frac{d}{dq}\left[\frac{f(q\mid a_L)}{f(q\mid a_H)}\right]\leq 0$$

This last inequality is precisely the MLRP condition. In the continuous-action case, the MLRP condition takes the following form:

$$\frac{d}{dq}\left[\frac{f_a(q\mid a)}{f(q\mid a)}\right]\geq 0$$

To get a monotonic transfer function, we must therefore make strong assumptions about $f(q \mid a)$. To obtain optimal *linear* incentive schemes, even stronger assumptions are required. This is unfortunate, as optimal linear incentive schemes are relatively straightforward to characterize. Also, in practice such schemes are commonly observed (as in sharecropping, for example). However, one also observes nonlinear incentive schemes like stock options for CEOs or incentive contracts for fund managers.

A final comment on the monotonicity of the transfer function: Perhaps a good reason why the function $w(q)$ must be monotonic is that the agent may be able to costlessly reduce performance, that is, "burn output". In the preceding example, he would then lower output from q_M to q_L whenever the outcome is q_M.

4.4.2 When Is the First-Order Approach Valid?

4.4.2.1 An Example where the First-Order Approach Is not Valid

We pointed out earlier that in general one cannot substitute the (IC) constraint with the agent's first-order condition (ICa). The following example due to Mirrlees (1975) provides an illustration of what can go wrong.[5] Mirrlees considers the following principal-agent example:

5. This example is admittedly abstract, but this is the only one to our knowledge that addresses this technical issue.

The principal's objective is to maximize

$$-(x-1)^2-(z-2)^2$$

with respect to z.

The agent, however, chooses x to maximize her objective:

$$u(x,z)=ze^{-(x+1)^2}+e^{-(x-1)^2}$$

For any z, the first-order condition of the agent's maximization problem is

$$z(x+1)e^{-(x+1)^2}+(x-1)e^{-(x-1)^2}=0$$

or

$$z=\frac{1-x}{1+x}e^{4x}$$

Now the reader can check that for z between 0.344 and 2.903 there are three values of x that solve this equation, one of which is the optimal value for the agent. To see which is the optimal value, observe that

$$u(z,x)-u(z,-x)=-(z-1)(e^{4x}-1)e^{-(x+1)^2}$$

Thus, for $z>1$ the maximum of u occurs for negative x. When $z<1$, the maximum of u occurs for positive x. In either case, this observation identifies the optimal value of x. When $z=1$, u is maximized by setting $x=0.957$ or $x=-0.957$.

By sketching indifference curves for the principal

$$(x-1)^2+(z-2)^2=K$$

in a diagram, one finds that the solution of the maximization problem of the principal is $x=0.957$ and $z=1$.

This solution is not obtained if one treats the problem as a conventional maximization problem with the agent's first-order condition as a constraint. One then obtains instead the following first-order conditions to the principal's problem:

$$(2-z)+(1-x)\frac{dx}{dz}=0$$

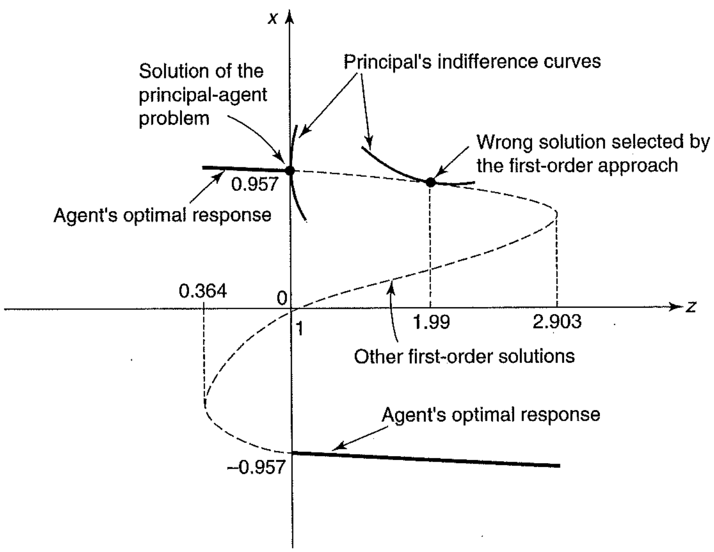

Figure 4.2
Solution of "Relaxed Problem" under the First-Order Approach

where

$$\frac{dz}{dx}=e^{4x}\left(\frac{4(1-x^2)-2}{(1+x)^2}\right)$$
$$=2z\left(\frac{2(1-x^2)-1}{(1+x)(1-x)}\right)$$

So that

$$2z(2-z)=\frac{(1-x^2)^2}{(1+x)(2x^2-1)}$$

There are three solutions x to this equation. One of those, defined by $z = 1.99$ and $x = 0.895$, achieves the highest value for the maximand

$$-(x-1)^2-(z-2)^2$$

but this solution does not maximize $u(x, z)$. This solution is only a local maximum.[6]

6. The second solution is ineligible on all possible grounds: it is a local minimum of $u(x, z)$. The third solution is a global maximum of $u(x, z)$ but does not maximize the principal's payoff.

Graphically the problem pointed out by Mirrlees can be represented as seen in Figure 4.2. The points on the bold curve represent the solutions to the agent's maximization problem. These points are global optima of the agent's problem. The point (x = 0.957; z = 1) is the optimum of the principal-agent problem. As the figure illustrates, by replacing the agent's (IC) constraint by only the first-order conditions of the agent's problem (ICa), we are in fact relaxing some constraints in the principal's optimization problem. As a result, we may identify outcomes that are actually not attainable by the principal (in this case, the point $x = 0.895$; $z = 1.99$).

4.4.2.2 A Sufficient Condition for the First-Order Approach to Be Valid

If the solution to the agent's first-order condition is unique and the agent's optimization problem is concave, then it is legitimate to substitute the agent's first-order condition for the agent's (IC).

Rogerson (1985a) gives sufficient conditions that validate this substitution: if MLRP, together with a convexity of the distribution function condition (CDFC) holds, then the first-order approach is valid. The CDFC requires that the distribution function $F(q|a)$ be convex in a:

$$F[q\,|\,\zeta a+(1-\zeta)a']\le \zeta F(q\,|\,a)+(1-\zeta)F(q,a')$$

for all $a, a' \in A$, and $\zeta \in [0,1]$.

The two conditions essentially guarantee that the agent's optimization problem is concave so that the first-order conditions fully identify global optima for the agent. The following heuristic argument shows why, under MLRP and CDFC, the agent's first-order conditions are necessary and sufficient. Suppose that the optimal transfer function $w(q)$ is differentiable almost everywhere.[7] The agent's problem is to maximize

$$\int_{\underline{q}}^{\overline{q}} u[w(q)]f(q\,|\,a)dq-\psi(a)$$

with respect to $a \in A$.

7. Note that since $w(q)$ is endogenously determined, there is no reason, a priori, for $w(q)$ to be differentiable.

Integrating by parts, we can rewrite the agent's objective function as follows:

$$u[w(\bar{q})]-\int_{\underline{q}}^{\bar{q}} u'[w(q)]w'(q)F(q\mid a)dq-\psi(a)$$

Differentiating this expression twice with respect to a, we obtain

$$-\int_{\underline{q}}^{\bar{q}} u'[w(q)]w'(q)F_{aa}(q\mid a)dq-\psi''(a)$$

Then, by MLRP $[w'(q) \geq 0]$ and CDFC $[F_{aa}(q\mid a) \geq 0]$, the second derivative with respect to a is always negative.

Unfortunately, CDFC and MLRP together are very restrictive conditions. For instance, none of the well-known distribution functions satisfy both conditions simultaneously. Jewitt (1988) has identified conditions somewhat weaker than CDFC and MLRP by making stronger assumptions about the form of the agent's utility function.

Notice, however, that the requirement that the agent's first-order conditions be necessary and sufficient is too strong. All we need is that the substitution of (IC) by its first-order condition yields necessary conditions for the principal's problem. These considerations suggest that an alternative approach to the problem that does not put stringent restrictions on the conditional distribution function $F(q|a)$ is useful. Such an approach has been developed by Grossman and Hart (1983a).

4.5 Grossman and Hart's Approach to the Principal-Agent Problem

This approach relies on the basic assumption that there are only a finite number of possible output outcomes. Thus, suppose that there are only N possible q_i: $0 \leq q_1 < q_2 < \ldots < q_N$, and let $p_i(a)$ denote the probability of outcome q_i given action choice a. The agent's action set A is taken to be a compact subset of $\mathcal{R}^n$. To keep the analysis simple we take the principal to be risk neutral: $V(\cdot) = q - w$. The agent's objective function, however, now takes the more general form

$$\tilde{u}(w,a)=\phi(a)u[w(q)]-\psi(a)$$

Under this specification, the agent's preferences over income lotteries are independent of her choice of action.[8] This utility function contains as special cases the multiplicatively separable utility function $[\psi(a) \equiv 0$ for all $a]$ and the additively separable utility function $[\phi(a) \equiv 1$ for all $a]$. Moreover, $u[w(q)]$ is such that $u(\cdot)$ is continuous, strictly increasing, and concave on the open interval $(\underline{w}, +\infty)$ and

$$\lim_{w\to\underline{w}} u(w)=-\infty$$

The principal's problem is to choose $a \in A$ and $w_i \equiv w(q_i) \in (\underline{w}, +\infty)$ to maximize

$$V=\sum_{i=1}^{N} p_i(a)(q_i-w_i)$$

subject to the agent's (IR) constraint

$$\sum_{i=1}^{N}\{\phi(a)u(w_i)-\psi(a)\}p_i(a)\geq\bar{u}$$

In addition, if a is unobservable the principal also faces the (IC) constraint

$$\sum_{i=1}^{N}\{\phi(a)u(w_i)-\psi(a)\}p_i(a)\geq\sum_{i=1}^{N}\{\phi(\hat{a})u(w_i)-\psi(\hat{a})\}p_i(\hat{a})$$

for all $\hat{a} \in A$.

Suppose, to begin with, that a is observable to the principal. Since the agent is risk averse and the principal risk neutral, the (first-best) optimal contract insures the agent perfectly. The optimal transfer is determined by maintaining the agent on her individual rationality constraint. Then the principal can implement the first best by setting

8. This is the most general representation of the agent's preferences such that the agent's participation (or IR) constraint is binding under an optimal contract. Under a more general representation, such that the agent's cost of effort depends on the agent's income or exposure to risk, it may be optimal for the principal to leave a monetary rent to the agent so as to lower the cost of effort and induce a higher effort choice. In that case the IR constraint would not be binding. As a consequence the optimal incentive contract would be substantially more difficult to characterize.