Microeconomic Theory-I
Washington State University
Midterm Exam #1 - *Answer key*

Fall 2016

1. **[Checking properties of preference relations].** Consider the following preference relation defined in the positive quadrant $X = \mathbb{R}^2_+$. A bundle (x_1, x_2) is weakly preferred to another bundle (y_1, y_2), i.e., $(x_1, x_2) \succsim (y_1, y_2)$, if and only if

$$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq \min\{3y_1 + 2y_2, 2y_1 + 3y_2\}$$

   (a) Consider bundle $(x_1, x_2) = (2, 1)$. For this bundle, draw the upper contour set (UCS), the lower contour set (LCS), and the indifference set (IND) of this preference relation.

   - Take a bundle $(2, 1)$. Then,

$$\min\{3 * 2 + 2 * 1, 2 * 2 + 3 * 1\} = \min\{8, 7\} = 7.$$

     The upper contour set of this bundle is given by

$$\begin{aligned} UCS(2,1) &= \{(x_1, x_2) \succsim (2,1)\} \\ &= \{\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq 7 \equiv \min\{8,7\}\} \end{aligned}$$

     which is graphically represented by all those bundles in $\mathbb{R}^2_+$ which are strictly above *both* lines $3x_1 + 2x_2 = 7$ and $2x_1 + 3x_2 = 7$. That is, for all (x_1, x_2) strictly above both lines

$$x_2 = \frac{7}{2} - \frac{3}{2}x_1 \text{ and } x_2 = \frac{7}{3} - \frac{2}{3}x_1.$$

     (See figure 1, which depicts these two lines and shades the set of bundles lying weakly above both lines.)

   - On the other hand, the lower contour set is defined as

$$\begin{aligned} LCS(2,1) &= \{(2,1) \succsim (x_1, x_2)\} \\ &= \{7 \geq \min\{3x_1 + 2x_2, 2x_1 + 3x_2\}\}, \end{aligned}$$

which is graphically represented by all bundles (x_1, x_2) weakly below the maximum of the lines described above. For instance, bundle $(y_1, y_2) = (2.5, 0)$, which lies on the horizontal axis and between both lines' horizontal intercept, implies

$$\min\{3 \cdot 2.5 + 2 \cdot 0, 2 \cdot 2.5 + 3 \cdot 0\} = \min\{7.5, 5\} = 5$$

thus implying that this consumer prefers bundle $(x_1, x_2) = (2, 1)$ than $(y_1, y_2) = (2.5, 0)$. A similar argument applies to all other bundles lying above $x_2 = \frac{7}{2} - \frac{3}{2}x_1$ and below $x_2 = \frac{7}{3} - \frac{2}{3}x_1$, where bundle $(2.5, 0)$ also belongs; see the triangle that both lines form at the right-hand side of the figure. Similarly, bundles such as $(0, 2.5)$ yield

$$\min\{3 \cdot 0 + 2 \cdot 2.5, 2 \cdot 0 + 3 \cdot 2.5\} = \min\{5, 7.5\} = 5,$$

which implies that the consumer also prefers bundle $(2, 1)$ to $(0, 2.5)$. An analogous argument applies to all bundles above line $x_2 = \frac{7}{2} - \frac{3}{2}x_1$ but below $x_2 = \frac{7}{3} - \frac{2}{3}x_1$ in the triangle at the left-hand side of Figure 1.

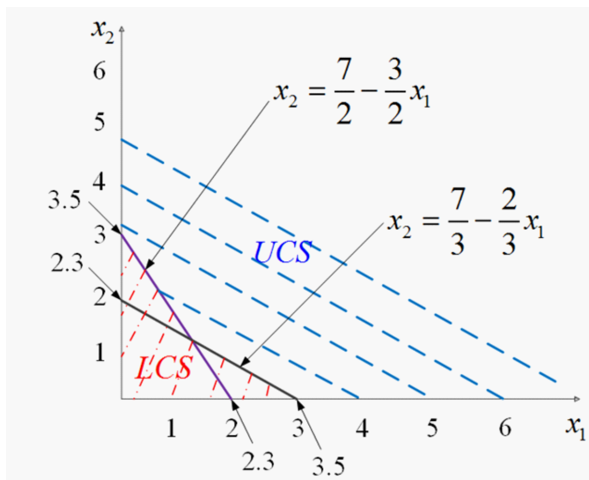

Figure 1. UCS and LCS of bundle (2,1).

Finally, those bundles for which the UCS and LCS overlap are those in IND of bundle (2,1).

(b) Show that this preference relation satisfies: (i) completeness, and (ii) transitivity.

- *Completeness.* First, note that both of the elements in the $\min\{\cdot\}$ operator are real numbers, i.e., $(3x_1 + 2x_2) \in \mathbb{R}_+$ and $(2x_1 + 3x_2) \in \mathbb{R}_+$, thus implying that the minimum

$$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} = a$$

exists and it is also a real number, $a \in \mathbb{R}_+$. Similarly, the minimum

$$\min\{3y_1 + 2y_2, 2y_1 + 3y_2\} = b$$

exists and is a real number, $b \in \mathbb{R}_+$. Therefore, we can easily compare a and b, obtaining that either $a \geq b$, which implies $(x_1, x_2) \succsim (y_1, y_2)$; or $a \leq b$, which implies $(y_1, y_2) \succsim (x_1, x_2)$, or both, $a = b$, which entails $(x_1, x_2) \sim (y_1, y_2)$. Hence, the preference relation is complete.

- *Transitivity.* We need to show that, for any three bundles (x_1, x_2), (y_1, y_2) and (z_1, z_2) such that

$$(x_1, x_2) \succsim (y_1, y_2) \text{ and } (y_1, y_2) \succsim (z_1, z_2), \text{ then } (x_1, x_2) \succsim (z_1, z_2)$$

First, note that $(x_1, x_2) \succsim (y_1, y_2)$ implies

$$a \equiv \min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq \min\{3y_1 + 2y_2, 2y_1 + 3y_2\} \equiv b$$

and $(y_1, y_2) \succsim (z_1, z_2)$ implies that

$$b \equiv \min\{3y_1 + 2y_2, 2y_1 + 3y_2\} \geq \min\{3z_1 + 2z_2, 2z_1 + 3z_2\} \equiv c$$

Combining both conditions we have that $a \geq b \geq c$, which implies that $a \geq c$. Hence, we have that

$$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq \min\{3z_1 + 2z_2, 2z_1 + 3z_2\}$$

and thus $(x_1, x_2) \succsim (z_1, z_2)$, implying that this preference relation is transitive.

2. [**Infering the utility function from the indirect utility.**] Consider an individual facing price vector $p = (p_1, p_2) >> 0$ and income $w > 0$. After solving his UMP, the individual finds that his indirect utility function is

$$v(p, w) = \left(p_1^\alpha p_2^{1-\alpha}\right) w.$$

Show that his utility function $u(x_1, x_2)$ must have a Cobb-Douglas representation.

[*Hint*: Use duality, where we learned that the indirect utility function evaluated at the expenditure function is $v(p, e(p, u)) = u$. Afterwards, use another result from duality: the Hicksian demand evaluated at $u = v(p, w)$ coincides with the Walrasian demand, i.e., $h_i(p, v(p, w)) = x_i(p, w)$ for every good $i = 1, 2$.]

- *Short proof.* From duality, the indirect utility function evaluated at $w = e(p, u)$ yields

$$v(p, e(p, u)) = u$$

which in this case entails $p_1^\alpha p_2^{1-\alpha} \cdot e(p, u) = u$. Solving for $e(p, u)$, we obtain

$$e(p, u) = \frac{u}{p_1^\alpha p_2^{1-\alpha}} = p_1 h_1(p, u) + p_2 h_2(p, u)$$

where the second equality is due to the definition of $e(p, u)$, and where $h_i(p, u)$ denotes the Hicksian demand of good $i = \{1, 2\}$. Differentiating $e(p, u)$ with respect to p_1 yields

$$\frac{\partial e(p, u)}{\partial p_1} = \alpha \frac{u}{p_1^{\alpha+1} p_2^{1-\alpha}} = h_1(p, u).$$

By duality, we know that the Hicksian demand evaluated at $u = v(p, w)$ coincides with the Walrasian demand, i.e., $h_i(p, v(p, w)) = x_i(p, w)$. Therefore, evaluating the above Hicksian demand at $u = v(p, w)$, where $w = e(p, u)$, yields

$$h_1(p, v(p, w)) = x_1(p, w) = \alpha \frac{p_1^\alpha p_2^{1-\alpha} w}{p_1^{\alpha+1} p_2^{1-\alpha}} = \alpha \frac{w}{p_1}.$$

And a similar argument applies if we differentiate $e(p,u)$ with respect to p_2, find the Hicksian demand for good 2, $h_2(p,u)$, and then evaluate it at $u = v(p,w)$ to obtain the Walrasian demand $x_2(p,w) = (1-\alpha)\frac{w}{p_2}$. These Walrasian demands are clearly of the Cobb-Douglas type, since the consumer spends a constant fraction of his income on each good. Then, $u(x_1,x_2) = x_1^{\alpha}x_2^{1-\alpha}$.

- *Long proof (EMP approach).* First, recall from duality that

$$u(x) \equiv \min_{p\in\mathbb{R}^2_+} v(p, p\cdot x) \tag{1}$$

Since $v(p,w)$ is homogeneous of degree zero, we can divide both arguments by $p\cdot x$ to obtain that the indirect utility function $v(p,w)$ is unaffected, i.e., $v(p,w) = v\left(\frac{p}{p\cdot x}, 1\right)$. Let $\overline{p} \equiv \frac{p}{p\cdot x}$, and thus $v(p,w) = v(\overline{p},1)$. As a consequence, if price vector p^* minimizes $v(p, p\cdot x)$ for an income level $p\cdot x = w$, then price vector $\overline{p}$ minimizes $v(p,1)$ for an income level $p\cdot x = 1$. That is, we can rewrite program (1) as

$$u(x) \equiv \min_{p\in\mathbb{R}^2_+} v(p,1) \quad \text{subject to} \quad p\cdot x = 1 \tag{2}$$

We can now find the price vector $p = (p_1,p_2)$ that solves program (2). Plugging them afterwards in the indirect utility function $v(p,1)$ will yield the original utility function $u(x)$ that this consumer maximized in his UMP (as stated in (2)). Since program (2) is a constrained minimization problem, we set up the Lagrangian

$$L = \left(p_1^{\alpha}p_2^{1-\alpha}\right) - \lambda(p_1x_1 + p_2x_2 - 1)$$

Taking first-order conditions with respect to p_1 and p_2 yields, respectively

$$\begin{aligned} \frac{dL}{dp_1} &= \alpha p_1^{\alpha-1}p_2^{1-\alpha} - \lambda x_1 = 0 \\ \frac{dL}{dp_2} &= (1-\alpha)p_1^{\alpha}p_2^{-\alpha} - \lambda x_2 = 0 \end{aligned}$$

and

$$\frac{dL}{d\lambda} = -p_1x_1 - p_2x_2 + 1 = 0$$

and simultaneously solving for p_1 and p_2 we obtain

$$p_1^* = \frac{\alpha}{x_1} \quad \text{and} \quad p_2^* = \frac{1-\alpha}{x_2}$$

We can finally plug these two prices, which solve (2), into the indirect utility funciton $v(p,1)$, yielding

$$\begin{aligned} v(p_1^*, p_2^*, 1) &= \left(\left(\frac{\alpha}{x_1}\right)^{\alpha}\left(\frac{1-\alpha}{x_2}\right)^{1-\alpha}\right)(1) \\ &= \alpha^{\alpha}(1-\alpha)^{1-\alpha}x_1^{-\alpha}x_2^{\alpha-1} \end{aligned}$$

which is clearly of the Cobb-Douglas type. For instance, labeling $A \equiv \alpha^{\alpha}(1-\alpha)^{1-\alpha}$ yields $v(p_1^*, p_2^*, 1) = Ax_1^{-\alpha}x_2^{\alpha-1}$, thus taking a more familiar format.

3. **[Finding the compensating and equivalent variation with little information.]** Consider a consumer who, facing a initial price vector $p^0 \in \mathbb{R}^n_{++}$ for n commodities, purchases a bundle $x \in \mathbb{R}^n_+$ with an income of $w > 0$ dollars. Assume that the price of all goods experiences a common increase measured by factor $\theta > 1$.

   (a) Compute the compensating variation (CV) of this price increase.

   - Using the expenditure function, the CV is

   $$CV = e(p^1, u^0) - e(p^0, u^0)$$

   where p^1 and p^0 denote the final and initial price vector, respectively, and u^0 represents the utility level that the consumer achieves at the initial price-wealth pair (p^0, w). In this exercise, we are informed that final prices p^1 satisfy $p^1 = \theta p^0$, thus implying that the above expression for CV can be rewritten as

   $$CV = e(\theta p^0, u^0) - e(p^0, u^0)$$

   Recall that the expenditure function is homogeneous of degree one in prices, i.e., $e(\theta p^0, u^0) = \theta e(p^0, u^0)$. In words, increasing the prices of all goods by a common factor θ increases the consumer's minimal expenditure (the expenditure he needs to reach utility level u^0) by exactly θ. In addition, the consumer spends w dollars, i.e., $e(p^0, u^0) = w$. These properties reduce the expression of the CV to

   $$\begin{aligned} CV &= e(\theta p^0, u^0) - e(p^0, u^0) = \\ &= \theta \underbrace{e(p^0, u^0)}_{w} - \underbrace{e(p^0, u^0)}_{w} = \\ &= \theta w - w = w(\theta - 1) \end{aligned}$$

   (b) Compute the equivalent variation (EV) of this price increase.

   - Using the expenditure function, the EV is

   $$EV = e(p^1, u^1) - e(p^0, u^1)$$

   where u^1 represents the utility level that the consumer achieves at the final price-wealth pair (p^1, w). In this exercise, we are informed that $p^1 = \theta p^0$, or $p^0 = \frac{1}{\theta} p^1$, implying that the above expression for EV can be rewritten as

   $$EV = e(p^1, u^1) - e\left(\frac{1}{\theta} p^1, u^1\right)$$

   Since the expenditure function is homogeneous of degree one in prices, i.e., $e\left(\frac{1}{\theta} p^1, u^1\right) = \frac{1}{\theta} e\left(p^1, u^1\right)$, and the consumer spends w dollars, $e(p^1, u^1) = w$. These properties reduce the EV to

   $$\begin{aligned} EV &= e(p^1, u^1) - e\left(\frac{1}{\theta} p^1, u^1\right) = \\ &= \underbrace{e(p^0, u^0)}_{w} - \frac{1}{\theta} \underbrace{e\left(p^1, u^1\right)}_{w} = \\ &= w - \frac{1}{\theta} w = w\left(1 - \frac{1}{\theta}\right) \end{aligned}$$

Following the same numerical example as in section (a), if all prices experience a 50% increase, i.e., $\theta = 1.5$, the equivalent variation would be $EV = 0.3w$, thus suggesting that, before the price increase, the consumer would need to give up a third of his wealth in order to be as worse off as he will be after the price increase.

(c) Evaluate your results from parts (a) and (b) if the common increase in prices is, specifically, $\theta = 1.5$ (that is, a 50% increase). Interpret the CV and EV in this specific case.

- For instance, increasing all prices by 50%, i.e., $\theta = 1.5$, yields a compensating variation of $CV = w(1.5 - 1) = 0.5w$, which implies that the consumer needs to receive half of his initial wealth in order to be able to reach the same utility level as before the price change.
- The equivalent variation would be $EV = w\left(1 - \frac{1}{1.5}\right) = 0.3w$, thus suggesting that, before the price increase, the consumer would need to give up a third of his wealth in order to be as worse off as he will be after the price increase.

4. [**Aggregation with additively separable preferences**] Consider a society with N individuals, and suppose that every individual $i \in N$ has preferences representable by the same additively separable utility function

$$u_i(x_1, x_2, ..., x_L) = \sum_{j=1}^{L} u_i(x_j)$$

where $u_i(x_j)$ denotes the utility that every individual i obtains from consuming x_j units of good $j = 1, 2, ..., L$. Assume that all functions $u_i(x_j)$ are (twice continuously) differentiable for every good j, and that marginal utility is strictly positive, i.e., $u_i'(x_j) > 0$ for every j. Assume that the price vector $p >> 0$ (i.e., it is positive in every component), and that every individual has the same wealth level $w > 0$. Last, assume that the utility maximization problem has a unique and interior solution. Show that we cannot guarantee that a positive representative consumer exists. That is, aggregate demand cannot be expressed as a function of p and aggregate wealth $\overline{w} = \sum_{i=1}^{N} w$ alone, or more compactly

$$\sum_{i=1}^{N} x_i(p, w) \neq x(p, \overline{w}).$$

[*Hint*: An example suffices.]

- No. Recall that, for a representative consumer to exist, the wealth expansion paths of all individuals in this economy must be straight and parallel to each other. We can find, however, several examples of separable utility functions whose wealth expansion paths are not straight. Here is one example:

$$u(x_1, x_2) = \sqrt{x_1} + \ln x_2.$$

Taking first order conditions with respect to x_1 and x_2, we obtain

$$\frac{1}{2\sqrt{x_1}} = \lambda p_1, \text{ and}$$

$$\frac{1}{x_2} = \lambda p_2$$

which implies

$$\frac{x_2}{2\sqrt{x_1}} = \frac{p_1}{p_2}$$

Solving for x_2, we find the equation $x_2 = 2\frac{p_1}{p_2}\sqrt{x_1}$, which represents the relationship between the utility-maximizing values of x_1 and x_2, i.e., the wealth expansion path. (Recall that wealth expansion path connects the utility maximizing bundles (x_1, x_2) for a given wealth level with those arising under a different wealth level.) Equation $x_2 = 2\frac{p_1}{p_2}\sqrt{x_1}$ is depicted in Figure 2.

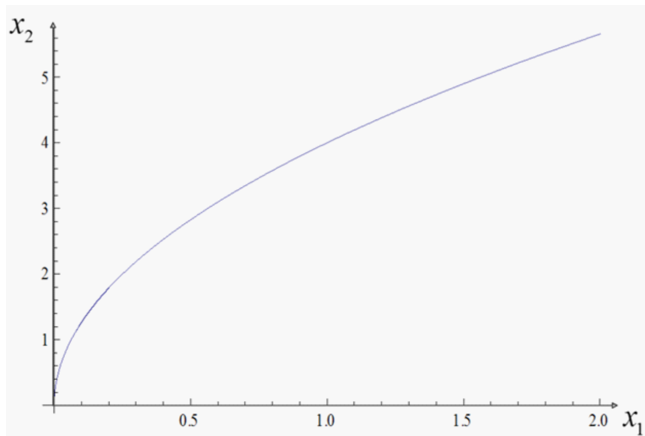

Figure 2. Wealth expansion path of $u(x_1, x_2) = \sqrt{x_1} + \ln x_2$.

The wealth expansion path, is, hence, not a straight line. Thus, no representative consumer exists.

- *Alternative approach.* An alternative way to show this result is by testing whether preferences are not homothetic: if they are not, the separable utility function will not generate a straight wealth expansion path. In particular, recall that to test for homotheticity, we need to check that the MRS is homogeneous of degree one. In this case,

$$MRS_{1,2}(x_1, x_2) = \frac{MU_1(x_1, x_2)}{MU_2(x_1, x_2)} = \frac{\frac{1}{2\sqrt{x_1}}}{\frac{1}{x_2}} = \frac{x_2}{2\sqrt{x_1}}.$$

And increasing all arguments by a common factor θ, the MRS becomes

$$MRS_{1,2}(\theta x_1, \theta x_2) = \frac{MU_1(\theta x_1, \theta x_2)}{MU_2(\theta x_1, \theta x_2)} = \frac{\theta x_1}{2\sqrt{\theta x_1}} = \sqrt{\theta}\frac{x_2}{2\sqrt{x_1}}$$

thus implying that

$$MRS_{1,2}(\theta x_1, \theta x_2) = \sqrt{\theta} MRS_{1,2}(x_1, x_2)$$

Finally, since $\sqrt{\theta}$ is not necessarily equal to 1, this separable utility function is not homothetic, thus not guaranteeing straight wealth expansion paths.

- Several popular preferences (such as the Stone-Geary, Cobb-Douglas, quasilinear for $L = 2$ goods, CES etc) can be represented by separable utility functions, and also by indirect utility functions of the Gorman form, for which a positive representative consumer exists. However, the representation of a consumer preference by an additively separable utility function is insufficient to guarantee the existence of a positive representative consumer, even if all consumers have identical utility functions.