Micro Theory I - EconS 501
Midterm #2 - *Answer key*

1. **[Deriving the cost function from the profit function]** Consider a profit-maximization problem (PMP) that produces the following profit function

$$\pi(p, w, r) = \frac{p^2}{4w} + \frac{p^2}{4r},$$

where $w \in \mathbb{R}_+$ denotes the wage rate, $r \in \mathbb{R}_+$ represents the interest rate of capital and $p \in \mathbb{R}_+$ denotes the price of the single output that the firm produces. Assume that w, r, and p are all exogenous, i.e., the firm's share in the product and input markets is negligible. Obtain the expression of its associated cost function $c(w, r, q)$.

- First, we can find he unconditional factor demand correspondences, $l(p, w, r)$ and $k(p, w, r)$, by applying Hotelling's lemma.[1]

$$\begin{aligned} l(p, w, r) &= \frac{\partial \pi(p, w, r)}{\partial w} = \frac{p^2}{4w^2}, \quad \text{and} \\ k(p, w, r) &= \frac{\partial \pi(p, w, r)}{\partial r} = \frac{p^2}{4r^2}. \end{aligned}$$

We can now obtain the supply correspondence $q(p, w, r)$ in a similar fashion

$$q(p, w, r) = \frac{\partial \pi(p, w, r)}{\partial p} = \frac{2p}{4w} + \frac{2p}{4r} = \frac{p(w + r)}{2wr}$$

And solving for p we obtain $p = \frac{2wrq}{w+r}$.

- We can now plug $p = \frac{2wrq}{w+r}$ into the *unconditional* factor demand correspondences in order to obtain the *conditional* factor demand correspondences, $z_l(w, r, q)$ and $z_k(w, r, q)$, as follows

$$\begin{aligned} z_l(w, r, q) &= \frac{\left(\frac{2wrq}{w+r}\right)^2}{4w^2} = \frac{r^2}{(w + r)^2} q^2, \quad \text{and} \\ z_k(w, r, q) &= \frac{\left(\frac{2wrq}{w+r}\right)^2}{4r^2} = \frac{w^2}{(w + r)^2} q^2 \end{aligned}$$

which we can finally use to find the cost function

$$\begin{aligned} c(w, r, q) &= w z_l(w, r, q) + r z_k(w, r, q) = \\ &= w \frac{r^2}{(w + r)^2} q^2 + r \frac{w^2}{(w + r)^2} q^2 = \frac{wr}{w + r} q^2 \end{aligned}$$

[1] Recall that unconditional demand correspondences, $l(p, w, r)$ and $k(p, w, r)$, do not depend on q; as opposed to the conditional factor demands, $z_l(w, r, q)$ and $z_k(w, r, q)$, which depend on the target output level q the firms seeks to reach.

- *Direct proof.* We first obtain the supply correspondence $q(p,w,r)$ as above

$$q(p,w,r) = \frac{\partial \pi(p,w,r)}{\partial p} = \frac{2p}{4w} + \frac{2p}{4r} = \frac{p(w+r)}{2wr}$$

In addition, we know that the profit function is

$$\pi(w,r,q(w,r,p)) = pq(p,w,r) - c(w,r,q(p,w,r))$$

Since $\pi(p,w,r) = \frac{p^2}{4w} + \frac{p^2}{4r}$ and $q(p,w,r) = \frac{p(w+r)}{2wr}$, the profit function becomes

$$\frac{p^2}{4w} + \frac{p^2}{4r} = p\frac{p(w+r)}{2wr} - c(w,r,q(p,w,r))$$

Solving for the cost function $c(w,r,q(p,w,r))$, we obtain

$$c(w,r,q(p,w,r)) = \frac{wr}{w+r}q^2$$

which coincides with $c(w,r,p)$ since by duality we know that $c(w,r,q(p,w,r)) = c(w,r,p)$.

2. [**Purchasing health insurance**] Consider an individual with the following utility function

$$u(C,H) = \ln C - \frac{\alpha}{H}$$

where C is his expenditure in consumption goods and H is his expenditure on health insurance. Parameter α denotes his monetary loss if he becomes sick where, for simplicity,

$$\alpha = \begin{cases} 1 \text{ if he is sick, and} \\ 0 \text{ if he is healthy} \end{cases}$$

Note that this utility function implies that, when getting sick, this individual's disutility decreases in the amount of health insurance that he purchased (e.g., he can have access to better doctors and care facilities, and the negative effects of the illness are reduced). The probability of getting sick is given by $\gamma \in [0,1]$, and this individual's wealth is given by $m > 0$, where $m = C + H$.

(a) What is this individual utility maximization problem? [*Hint:* Rearrange the individual's expected utility maximization problem so his only choice variable is C.]

- The individual chooses C and H to solve

$$\begin{aligned} & \max_{C,H} \; (1-\gamma)\ln C + \gamma\left(\ln C - \frac{1}{H}\right) \\ \text{subject to } m &= C + H \end{aligned}$$

where we substitute $\alpha = 0$ when the individual is healthy (which occurs with probability $1-\gamma$), and $\alpha = 1$ when the individual is sick (which occurs with probability γ). This maximization problem can nevertheless be simplified.

In particular, since $m = C + H$, i.e., $H = m - C$, the expected utility maximization problem becomes

$$\max_{C} \; (1-\gamma)\ln C + \gamma\left(\ln C - \frac{1}{m-C}\right)$$

which reduces the choice variables of this maximization problem to only one: C.

(b) Find the first order conditions associated to the previous maximization problem.

- Taking first order condition with respect to C,

$$\frac{1}{C} - \frac{\gamma}{(m-C)^2} = 0$$

(c) Determine the optimal amount of consumption goods, C^*, and health insurance, H^*.

- Rearranging the above first order condition, we obtain

$$C^2 - (2m+\gamma)C + m^2 = 0$$

with solutions

$$C = \frac{2m+\gamma+\sqrt{\gamma^2+4m\gamma}}{2} \text{ and } C = \frac{2m+\gamma-\sqrt{\gamma^2+4m\gamma}}{2}$$

Given that the amount spent on consumption cannot exceed the individual's wealth, $C \leq m$, the only feasible solution is $C^* = \frac{2m+\gamma-\sqrt{\gamma^2+4m\gamma}}{2}$. (Indeed, note that the alternative root, $\frac{2m+\gamma+\sqrt{\gamma^2+4m\gamma}}{2}$, is unambiguously larger than the individual's wealth, m.) Therefore, the optimal amount of health insurance that this individual buys is

$$H^* = m - C^* = m - \frac{2m+\gamma-\sqrt{\gamma^2+4m\gamma}}{2} = \frac{\sqrt{\gamma^2+4m\gamma}-\gamma}{2}$$

(d) Determine if the optimal amount of health insurance, H^*, is increasing, decreasing, or constant in m. Interpret.

- Differentiating H^* with respect to m,

$$\frac{\partial H^*}{\partial m} = \frac{\gamma}{\sqrt{\gamma + 4m\gamma}}$$

which is positive for all parameter values. That is, the optimal amount of health insurance, H^*, is increasing in the individual's wealth level, m.

3. [**Cost-reducing investment in monopoly**] Consider a monopolist with inverse demand function $p(q) = a - bq$. The monopolist makes two choices: how much to invest in cost reduction, A, and how much to produce, q. If the monopolist invests A units in cost reduction, his (constant) per-unit cost of production is $c(A) = c - \beta\sqrt{A}$, where

$c > 0$ is the initial marginal cost, and β denotes the effectiveness of cost-reducing investment. This implies that

$$c'(A) = -\frac{\beta}{2\sqrt{A}} < 0 \quad \text{and} \quad c''(A) = \frac{\beta}{4A^{\frac{3}{2}}} > 0,$$

(i.e., investing in cost reduction decreases the monopolist's per-unit cost of production, but at a decreasing rate; as depicted in figure 1 for two values of parameter β, where $\beta_2 > \beta_1$.) For simplicity, you can assume that $a > c$, and $b > \frac{\beta^2}{2}$.

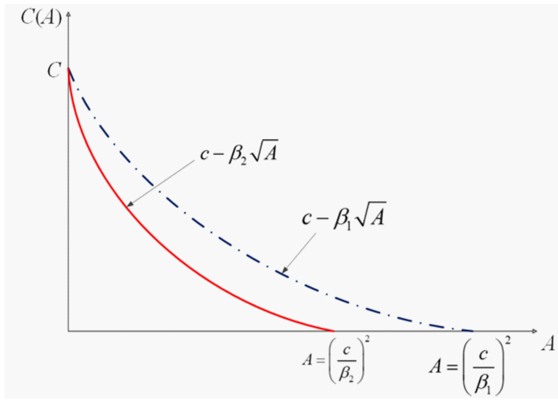

Figure 1. Cost function $c(A)$.

(a) *Unregulated monopolist.* Derive the first-order conditions for the monopolist's choices.

- The monopolist will solve

$$\max_{q,A} \quad (a - bq) \cdot q - (c - \beta\sqrt{A})q - A$$

Taking first order condition with respect to q yields

$$a - 2bq^m - (c - \beta\sqrt{A^m}) = 0$$

and taking first order condition with respect to A, we obtain

$$\frac{\beta}{2\sqrt{A^m}}q^m - 1 = 0.$$

Simultaneously, solving for q^m and A^m in the above first order conditions yields

$$q^m = \frac{2(a-c)}{4b - \beta^2} \quad \text{and} \quad A^m = \frac{(a-c)^2\beta^2}{(4b - \beta^2)^2}.$$

which is positive for all $b > \frac{\beta^2}{4}$, which holds given the assumption $b > \frac{\beta^2}{2}$ where $\frac{\beta^2}{2} > \frac{\beta^2}{4}$.

- *Ineffective investment,* $\beta = 0$. When $\beta = 0$, the monopolist produces the standard output level $q^m = \frac{a-c}{2b}$ and does not invest on cost-reducing technologies, i.e., $A^m = 0$.

- *Effective investment,* $\beta > 0$. When investing in cost-reducing technologies is effective, the monopolist invests a positive amount in cost-reducing technologies, i.e., $A^m = \frac{(a-c)^2\beta^2}{(4b-\beta^2)^2} > 0$, which reduces its production costs ultimately yielding a larger output level, i.e., q^m increases in the effectiveness of cost-reducing investments, β, since

$$\frac{\partial q^m}{\partial \beta} = \frac{4\beta(a-c)}{(4b-\beta^2)^2}$$

is positive for all parameter values.

(b) *First best.* Compare the monopolist's choices with those of a benevolent social planner who can control both q and A (a "first-best" comparison). Interpret your results.

- The social planner will maximize total surplus,

$$\max_{q,A} \int_0^q (a - bx)dx - (c - \beta\sqrt{A})q - A,$$

Taking first order condition with respect to q yields

$$a - bq - (c - \beta\sqrt{A}) = 0$$

and taking first order condition with respect to A, we obtain

$$\frac{\beta}{2\sqrt{A}}q - 1 = 0.$$

Simultaneously solving for q^{sp} and A^{sp} yields

$$q^{sp} = \frac{2(a-c)}{2b-\beta^2} \quad \text{and} \quad A^{sp} = \frac{(a-c)^2\beta^2}{(2b-\beta^2)^2}.$$

which is positive given that $b > \frac{\beta^2}{2}$ by assumption.

- *Ineffective investment,* $\beta = 0$. If cost-reducing technologies are ineffective, $\beta = 0$, then the social planner produces the standard socially optimal output level $q^{sp} = \frac{a-c}{b}$ and invests nothing $A^{sp} = 0$.
- *Effective investment,* $\beta > 0$. In this case, both q^{sp} and A^{sp} increase in β. Moreover, comparing A^{sp} with A^m, we can notice that $A^{sp} > A^m$, suggesting that the monopolist invests less in cost-reducing technologies than the social planner would. In addition, $q^{sp} > q^m$, also indicating that the monopolist production, while increasing in β, is still socially insufficient.

(c) *Second best.* Assume now that the social planner can control for the investment in cost-reducing technologies, A, but *not* for q (i.e., he can implement a "second-best" policy). In particular, suppose that the social planner chooses A and afterwards the monopolist responds choosing q. Compare your results with those in part (b) where the regulator, choosing both A and q, implements a first-best policy.

- Given a level $\widehat{A}$ set by the government, the monopolist will set q to maximize its profits, i.e., it will set q to equate $MR = MC$. Therefore, the governments problem is to maximize social surplus subject to the monopolists's behavior. That is,

$$\max_{q,A} \quad \int_0^q (a - bx)dx - (c - \beta\sqrt{A})q - A$$
$$\text{subject to} \;\; a - 2bq = c - \beta\sqrt{A}$$

The Lagrangian is

$$\mathcal{L} = \int_0^q (a - bx)dx - (c - \beta\sqrt{A})q - A - \lambda[a - 2bq - (c - \beta\sqrt{A})]$$

which yields the first order conditions of

$$\frac{\partial \mathcal{L}}{\partial q} = a - b\widehat{q} - (c - \beta\sqrt{A}) + 2b\lambda = 0,$$

$$\frac{\partial \mathcal{L}}{\partial A} = \frac{-2\sqrt{\widehat{A}} + \beta(\widehat{q} - \lambda)}{2\sqrt{\widehat{A}}} = 0, \quad \text{and}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = a - 2b\widehat{q} - c + \beta\sqrt{\widehat{A}} = 0$$

Simultaneously solving for $\widehat{q}$ and $\widehat{A}$ yields

$$\widehat{q} = \frac{4(a-c)}{8b - 3\beta^2} \quad \text{and} \quad \widehat{A} = \frac{(a-c)^2\beta^2}{(8b - 3\beta^2)^2}.$$

which is positive for all $b > \frac{3\beta^2}{8}$, which holds given the assumption $b > \frac{\beta^2}{2}$ where $\frac{\beta^2}{2} > \frac{3\beta^2}{8}$.

- *Comparing first- and second-best policies.* Comparing $\widehat{A}$ and A^{sp}, we see that $\widehat{A} > A^{sp}$. indicating that, in this second-best policy, the social planner needs to select a larger cost-reducing investment in order to induce the monopolist to produce an output level $\widehat{q}$ closer to the social optimal q^{sp} (but still suboptimal). Figure 2 illustrates A^{sp} (first best) and $\widehat{A}$ (second best) for parameter values $a = b = 1$ an $c = 0$. (Other parameter values yield similar results.)

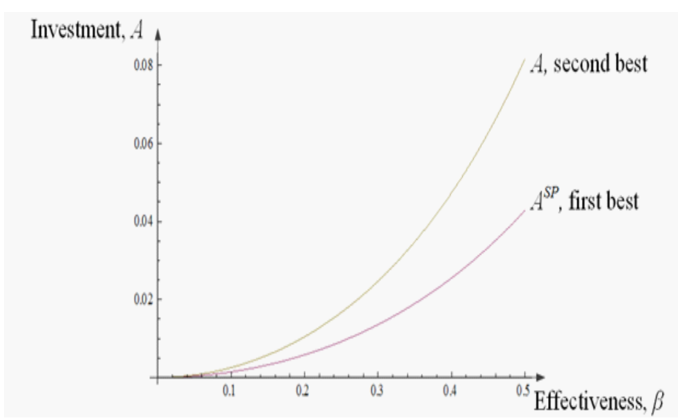

Figure 2. First- and second-best investments.

Comparing output levels $\widehat{q}$ and q^{sp}, we find

$$q^{sp} - \widehat{q} = \frac{2(a-c)}{2b-\beta^2} - \frac{4(a-c)}{8b-3\beta^2} = \frac{2(a-c)(4b-\beta^2)}{16b^2+3\beta^4-14b\beta^2}$$

which is positive for all $b > \frac{\beta^2}{4}$, which holds given the assumption $b > \frac{\beta^2}{2}$ where $\frac{\beta^2}{2} > \frac{\beta^2}{4}$. Figure 3 depicts the first-best output, q^{sp}, and the second-best output, $\widehat{q}$. (For consistency, we use the same parameter values as in figure 2.)

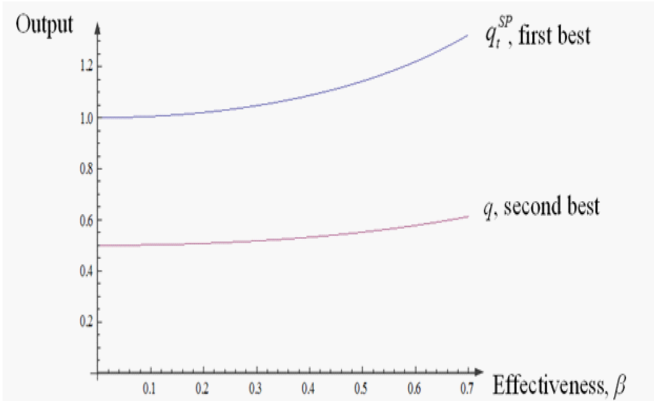

Figure 3. First- and second-best output.