ECONS 301 – INTERMEDIATE MICROECONOMICS

Midterm #2 – *Answer key*

Exercise #1 – Consumer theory for goods regarded as substitutes

Consider a consumer with the following linear utility function for two goods, 1 and 2, which are regarded as substitutes by the consumer

$$u(x_1, x_2) = 2x_1 + x_2$$

Assume that the consumer faces a price of \$1 for good 2, and a total income of \$*I*. The price of good 1 is left unrestricted as p_1.

a) Find the marginal rate of substitution, *MRS*, for this consumer.

Answer:

$$MRS = \frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2} = \frac{MU_1}{MU_2} = 2$$

b) *Utility maximization problem.* Set up this consumer's utility maximization problem (UMP), and find the Walrasian demand.

Answer: We first write up the tangency condition as follows:

$$\frac{MU_1}{MU_2} = 2 = \frac{p_1}{p_2} = p_1$$

Or

$$p_1 = 2$$

Since the consumer regards goods 1 and 2 as perfect substitutes, we have that "bang for the buck" of good 1, $\frac{MU_1}{p_1} = \frac{2}{p_1}$, exceeds that of good 2, $\frac{MU_2}{p_2} = \frac{1}{1} = 1$, if $\frac{2}{p_1} > 1$. Solving for p_1, we obtain that this condition holds when p_1 satisfies $p_1 < 2$. In this case, the individual only consumes positive units of good 1, that is, $p_1 x_1^* = I$ or $x_1^* = \frac{I}{p_1}$ and $x_2^* = 0$. If, instead, the price of good 1 satisfies $p_1 \geq 2$, (For simplicity, we can assume that when $p_1 = 2$ the consumer picks the same optimal consumption bundle as when $p_1 > 2$.) the consumer only purchases units of good 2, that is, $x_2^* = I$ and $x_1^* = 0$.

c) Solve for income *I*, in order to obtain the Engel curve of good 1. Is the slope of the Engel curve positive? Interpret: is the good normal or inferior?.

Answer: If p_1 satisfies $p_1 < 2$, given the demand function found in part b, the Engel curve of good 1 is:

$$I = p_1 x_2$$

Thus, the slope of the Engel curve is positive, implying it is a normal good. Since in this case the consumer does not purchase units of good 2, the Engel curve of good 2 does not exist.

If, instead, the price of good 1 satisfies $p_1 \geq 2$, given the demand function found in part b, the Engel curve of good 2 is:

$$I = x_2$$

Thus, the slope of the Engel curve is positive, implying that this is a normal good. Similarly, the Engel curve of good 1 does not exist since the consumer does not purchase any unit of good 1.

d) *Expenditure minimization problem.* For the remainder of the exercise, you can assume an income of I=\$100. Set up this consumer's expenditure minimization problem (EMP), assuming that he seeks to reach a target utility level of $\bar{u}$. Find the Hicksian demand (also referred to as the "compensated" demand).
[*Hint*: Write the tangency condition, solve for x_2, and insert your result into the consumer's utility function. Solving for x_1, you will obtain the demand for good 1. Recall that the expression you find should be a function of the price of good 1, p_1.]
Answer: We first write up the tangency condition as follows:

$$\frac{MU_1}{MU_2} = 2 = \frac{p_1}{p_2} = p_1$$

Or

$$p_1 = 2$$

Since the consumer regards goods 1 and 2 as perfect substitutes, we have that "bang for the buck" of good 1, $\frac{MU_1}{p_1} = \frac{2}{p_1}$, exceeds that of good 2, $\frac{MU_2}{p_2} = \frac{1}{1} = 1$, if p_1 satisfies $p_1 < 2$. In this case, the individual only consumes positive units of good 1, that is, $2x_1^e = \bar{u}\ or\ x_1^e = \frac{\bar{u}}{2}$ and $x_2^e = 0$. If, instead, the price of good 1 satisfies $p_1 \geq 2$ the consumer only purchases units of good 2, that is, $x_2^e = \bar{u}$ and $x_1^e = 0$.[1]

e) Assume now that the price of good 1 decreases from $p_1 = \$4$ to $p_1 = \$2$. Find the increase in consumer surplus that this consumer enjoys from the price decrease.
Answer: Since the price of good 1 decreases from $p_1 = \$4$ to $p_1 = \$2$, the situation falls to the case that $p_1 \geq 2$. Given the perfect substitutes preference, the individual will still spend all the income for good 2. Therefore, the CS doesn't change at all.

f) Considering the same price decrease as in part (e), find the compensating variation (CV).
Answer: Since the price of good 1 decreases from $p_1 = \$4$ to $p_1 = \$2$, the situation falls to case that $p_1 \geq 2$. So The individual will consume only good 2. Recall that

CV= Cost of bundle A – Cost of bundle B.

With initial inome,we have:

$$x_1 = \frac{100}{1} = 100 \text{ and } x_2 = 0$$

Which is the bundle A with initial prices. Which is the bundle A with initial prices. To find bundle B, we first need to determine the utility level of bundle A

$$U_A = 100 = 2x_1^B$$

In addition, bundle B has the indifference curve being tangent to the decomposition budget line (in this case indifference curve overlap with the budget constraint and the individual only purchase good 1). Then, we can obtain bundle B,

$$x_1^B = 50 \text{ and } x_2^B = 0$$

implying that the income that the individual needs to purchase bundle *B* is

$$I_B = 50 * 2 = 100.$$

Therefore, we obtain compensating variation, that is:

$$CV = I_A - I_B = 100 - 100 = 0$$

g) Considering the same price decrease as in part (e), find the equivalent variation (EV).

[1] For simplicity, we can assume that when $p_1 = 2$ the consumer picks the same optimal consumption bundle as when $p_1 > 2$.

Answer: Since the price of good 1 decreases from $p_1 = \$4$ to $p_1 = \$2$, the situation falls to case that $p_1 > 2$. So the individual will only consume positive units of good 2. Recall that

EV= Cost of bundle E – Cost of bundle C.

With final price, we have:

$$x_1 = \frac{100}{2} = 50 \text{ and } x_2 = 0$$

which is the bundle C with the price changed. To find bundle E, we know that the individual reaches the same utility level as at bundle C, that is,

$$U_C = 50 * 2 = 100 = 4x_1^E$$

Furthermore, bundle E must have that the indifference curve is tangent to the budget line at the initial prices(in this case indifference curve overlap with the budget constraint and the individual only purchase good 1), that is,

$$x_1^E = 25 \text{ and } x_2^E = 0$$

Hence the income that the individual needs to purchase bundle *E* is

$$I_E = 25 * 4 = 100$$

Therefore, we obtain equivalent variation, that is:

$$EV = I_E - I_C = 100\text{-}100\text{=}0$$

Exercise #2 – Finding average and marginal products

Consider a production function

$$f(k,l) = \left[243 + \tfrac{1}{3}(l-9)^3\right]k,$$

where k represents units of capital and l denotes units of labor.

a) Fix the amount of capital at $k = 1$. Calculate the average product and marginal product (when only labor can be varied).
b) Now fix the amount of labor at $l = \bar{l}$. Calculate the average product and marginal product (when only capital can be varied).
c) Compare your results in parts (a) and (b), and describe how each input plays a different role in this firm.

Answer:

a) When the amount of capital is fixed at *k*=1, total product is

$$f(1,l) = [243 + \tfrac{1}{3}(l-9)^3].$$

The average product of labor is then

$$AP_l(1,l) = \frac{f(1,l)}{l} = \frac{243+\frac{1}{3}(l-9)^3}{l},$$

and the marginal product of labor is

$$MP_l(1,l) = \frac{\partial f(1,l)}{\partial l} = \frac{1}{3} * 3 * (l-9)^2 = (l-9)^2.$$

b) When the amount of labor is fixed at $l = \bar{l},$ total product is

$$f(k,\bar{l}) = [243 + \tfrac{1}{3}(\bar{l}-9)^3]k.$$

The average product of capital is

$$AP_k(k,\bar{l}) = \frac{f(k,\bar{l})}{k} = \frac{[243+\frac{1}{3}(\bar{l}-9)^3]k}{k} = 243 + \frac{1}{3}(\bar{l}-9)^3,$$

and the marginal product of labor is

$$MP_k(k,\bar{l}) = \frac{\partial f(k,\bar{l})}{\partial k} = 243 + \frac{1}{3}(\bar{l}-9)^3.$$

c) In part (a), $AP(l, k)$ and $MP(l, k)$ vary with the amount of labor, because AP and MP are functions of l.
In contrast, in part (b), $AP(k, \bar{l})$ and $MP(k, \bar{l})$ coincide, and are both independent on the amount of capital, k (note that capital doesn't show up in the expressions of $AP(k, \bar{l})$ and $MP(k, \bar{l})$ that we found on part b). Hence, both average product and marginal product curves are constant (flat lines) for a given amount of labor ($\bar{l}$).

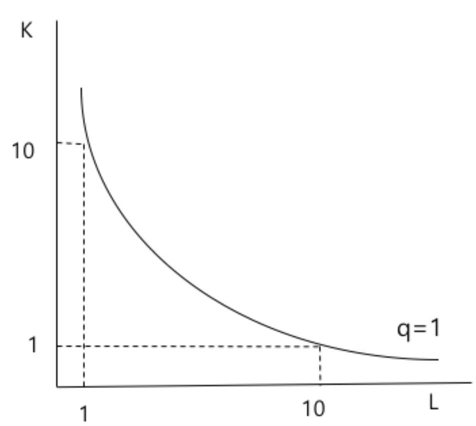
Exercise #3 – Finding labor and capital demands for the firm

Consider a firm producing prosciutto (a dried Italian ham nowadays not recommended by the World Health Organization) with production function $q = f(k, l) = \frac{k \cdot l}{10}$, that faces input prices $w = \$10$ and $r = \$100$ for labor and capital, respectively.

a) Find the isoquant of an output $q = 1$. [*Hint*: Set $q = 1$, and then solve for k.] Draw it in a figure with l in the horizontal axis and k in the vertical axis.
b) Repeat your steps in part (a) in order to find the isoquant corresponding to $q = 2$, and the isoquant corresponding to $q = 3$.
c) Does this firm's production exhibit increasing, decreasing or constant returns to scale?
d) Find the labor demand, and the capital demand, as a function of q.
e) Find the firm's long-run cost function $TC(q)$.
f) If the firm wanted to produce only one unit of prosciutto, how many units of labor and capital should it use? How much will it cost? What if the firm seeks to produce two units?
g) Find the firm's long-run average cost function, $AC(q)$, and its long-run marginal cost function, $MC(q)$. Graph $AC(q)$ and $MC(q)$, and identify the firm's long-run supply curve.

Answer:

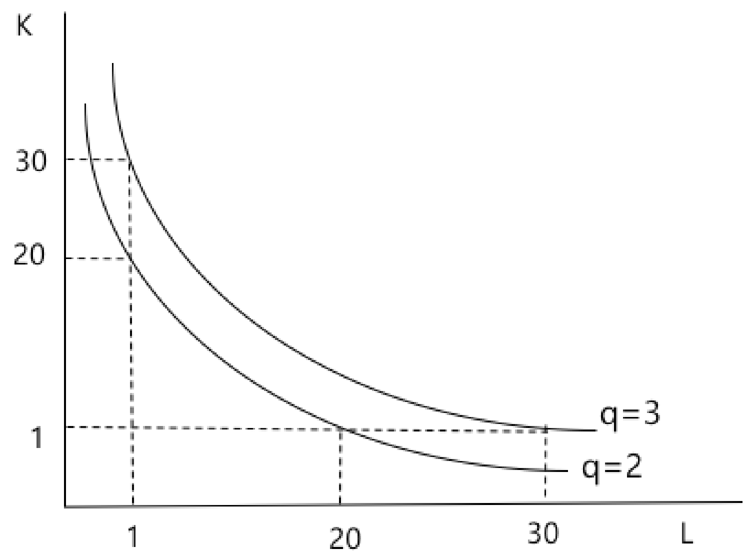
a) For a given output q=1, we have $1 = f(k, l) = \frac{k*l}{10}$. That is, $1 = \frac{k*l}{10}$ which we can solve for k to obtain the equation of the isoquant, $k = \frac{10}{l}$. This isoquant is depicted in the next figure. When $l \to 0$, the isoquant $k = \frac{10}{l}$ approaches infinity, as depicted in the left-hand side of the figure. When $l \to \infty$, the isoquant $k = \frac{10}{l}$ approaches zero, as illustrated in the right-hand side of the figure.



b) Since $q = \frac{k*l}{10}$, we can solve for k in order to find the general expression of the isoquant for any output level q,

$$k = \frac{10 * q}{l}.$$

Hence, the isoquant corresponding to an output level of q=2 is $k = \frac{10*2}{l} = \frac{20}{l}$, and the isoquant corresponding to an output of q=3 is $k = \frac{10*3}{l} = \frac{30}{l}$.



c) Increasing all inputs by a common factor λ yields

$$f(\lambda k, \lambda l) = \frac{(\lambda k)(\lambda l)}{10} = \lambda^2 \frac{k*l}{10} = \lambda^2 f(k, l).$$

Because $\lambda^2 > \lambda$, this firm's production shows increasing returns to scale. Intuitively, output responds more than proportionally to a given increase in all inputs. If, for example, all inputs are doubled, $\lambda = 2$, then output responds by increasing $\lambda^2 = 2^2 = 4$ times.

d) From the tangency condition between isoquant and isocost, we obtain that

$$\frac{MP_l}{MP_k} = \frac{k/10}{l/10} = \frac{k}{l} = \frac{10}{100} = \frac{w}{r},$$

which can be rearranged as $l = 10k$.

- Plugging this equation into the production function, we obtain that $q = \frac{k*10k}{10} = k^2$. Thus, solving for k, the capital demand function can be represented as $k = \sqrt{q}$.
- Plugging the equation, we found from the tangency condition, $l = 10k$ or $k = \frac{l}{10}$, into the production function, we obtain that $q = \frac{(l/10)*l}{10} = \frac{l^2}{100}$. Therefore, solving for l, we find that the labor demand function is $l = 10\sqrt{q}$.

e) The firm's long-run total cost function is that $C(q) = rk + wl$. Since $r = 100, w = 10$, $k = \sqrt{q}$, and $l = 10\sqrt{q}$,

$$C(q) = \left[100 * \sqrt{q}\right] + [10 * \left(10\sqrt{q}\right)].$$

f) From the result of part (d), when q=1, $k = \sqrt{q} = 1$ and $l = 10\sqrt{q} = 10$, so that the cost of capital and labor become $rk = 100 * 1 = 100$, and $wl = 10 * 10 = 100$, respectively. When the firm seeks to produce two units, q=2, capital demand is $k =$

$\sqrt{q} = \sqrt{2}$ and $l = 10\sqrt{q} = 10\sqrt{2}$, so that the cost of capital becomes $rk = 100 * \sqrt{2} = 100\sqrt{2}$, and the cost of labor is $wl = 10 * 10\sqrt{2} = 100\sqrt{2}$.

g) From the result of part (e), the firm's long-run total cost function is that

$$C(q) = \left[100 * \sqrt{q}\right] + [10 * (10\sqrt{q})].$$

Therefore, the average cost function is

$$AC(q) = \frac{C(q)}{q} = \frac{100\sqrt{q}+100\sqrt{q}}{q} = \frac{200\sqrt{q}}{q} = \frac{200}{\sqrt{q}},$$

and the marginal cost function is

$$MC(q) = \frac{\partial C(q)}{\partial q} = 100q^{-\frac{1}{2}} = \frac{100}{\sqrt{q}}.$$

Both AC and MC curves are decreasing in output q, and MC<AC. Therefore, there is no long-run supply curve.

Exercise #4 – Perfectly competitive markets

Consider a good *x* that is only produced in France. There are 10,000 small firms producing good *x*, and they all use a Cobb-Douglas production function

$$f(k,l) = k^{1/3}l^{2/3},$$

and all firms face input prices $w = \$1$ and $r = \$256$.

a) Find the long-run $, MC$ and $, AC$ curves for each of these firms.
b) What is the long-run market (or aggregate) supply for the whole industry?
c) Assume that good *x* is only consumed in Germany, with a demand function $x(p) = \frac{36{,}000}{p}$. Using the market supply you found in part (b), find the competitive equilibrium price, the equilibrium quantity of *x*, each firm's output, and each firm's profits.
d) If the demand for *x* in Germany changes to $x(p) = \frac{24{,}000}{p}$, how do your results in part (c) change?

Answer:

a) We use the tangency condition between the firm's isoquant and isocost, $\frac{MP_l}{MP_k} = \frac{w}{r}$, to obtain the optimal amount of labor and capital that the firm hires. First, we find the marginal products of each input, as follows

$$MP_l = \frac{\partial f(k,l)}{\partial l} = \frac{2}{3} * k^{1/3}l^{-1/3} \text{ and } MP_k = \frac{\partial f(k,l)}{\partial k} = \frac{1}{3} * k^{-2/3}l^{2/3},$$

Hence, the ratio of marginal products is

$$\frac{MP_l}{MP_k} = \frac{2k^{1/3}}{3l^{1/3}} * \frac{3k^{2/3}}{l^{2/3}} = \frac{2k}{l}.$$

Therefore, we can solve $\frac{2k}{l} = \frac{w}{r} = \frac{1}{256}$, which can be rearranged into $l = 512k$ or $k = \frac{l}{512}$. By plugging this equation into the production function $f(k,l) = k^{1/3}l^{2/3}$, we can get

$$q = k^{1/3} * (512k)^{2/3} = 64k \text{ or } k = \frac{q}{64},$$

And, similarly,

$$q = \left(\frac{l}{512}\right)^{1/3} * l^{2/3} = \frac{l}{8} \text{ or } l = 8q.$$

Now, we can now use the above demand for capital, $k = \frac{q}{64}$, and for labor, $l = 8q$, to calculate the total cost of production,

$$TC = wl + rk = (1 * 8q) + \left(256 * \frac{q}{64}\right) = 12q.$$

Thus, marginal costs are

$$MC = \frac{\partial TC}{\partial q} = \frac{\partial(12q)}{\partial q} = 12,$$

and average costs are

$$AC = \frac{TC}{q} = \frac{12q}{q} = 12.$$

b) The individual firm's long-run market supply is determined at $MC = 12$.

c) From the result of (b), we know the competitive equilibrium price is 12, and therefore $x(p) = \frac{36,000}{12} = 3000$. Since there are 10,000 firms in the whole economy, each firm's output is $\frac{3,000}{10,000} = 0.3$ units, and each firm's profit is

$$\pi = TR - TC = px - wl - rk = px - 12x = (12 * 0.3) - (12 * 0.3) = 0$$

d) we know the competitive equilibrium price is 12, and therefore $x(p) = \frac{24000}{12} = 2000$. Since there are 10,000 firms in the whole economy, each firm's output is $\frac{2000}{10000} = 0.2$ units, and each firm's profit is

$$\pi = TR - TC = px - wl - rk = px - 12x = (12 * 0.2) - (12 * 0.2) = 0$$

Exercise #5 - Monopoly

Assume that a firm is the unique producer in a market. This monopolist faces a demand curve $p(q) = 210 - 4q$ and initially faces a constant marginal cost $MC = 5$.

a) Calculate the profit-maximizing quantity for this monopolist.
b) What will be this monopolist's optimal price?
c) What is the monopolist's total revenue at that price?
d) Suppose that the monopolist's marginal cost increases to $MC = 20$. What is the monopolist optimal quantity now?
e) What will be the monopolist's optimal price?
f) What is the monopolist's total revenue at the price you found in part (e)?

Answer:

a) A monopolist chooses an optimal production level at $MR = MC$. In this case,

$$MR = \frac{\partial(PQ)}{\partial Q} = \frac{\partial[(210 - 4Q) * Q]}{\partial Q} = 210 - 8,$$

and , $MC = 5$. Hence, setting , $MR = MC$ yields

$$210 - 8Q = 5,$$

and solving for output Q, we obtain $205 = 8Q$, or $Q = \frac{205}{8} = 25.6$ units.

b) The monopolist's price is $p = 210 - 4Q = 210 - 4(25.6) = \$107.5$.

c) Total revenue is $TR = pQ = 107.5 * 25.6 = \$2,754.69$.

d) If the marginal cost increases to $MC = 20$, the monopoly sets $MR = MC$ which in this case yields

$$MR = 210 - 8Q = 20 = MC,$$

which can be solved as $190 = 8Q$ or $Q = \frac{190}{8} = 20$ units. Hence, when MC increases, the optimal quantity decreases from 25.6 to 20 units.

e) The optimal price is $p = 210 - 4(20) = \$130$.

f) The total revenue is $TR = pQ = 130 * 20 = \$2,600$, since $p = \$130$.

Exercise #6 - Multiplant monopoly

Consider a monopoly operating two plants, 1 and 2, with total costs $TC_i(q_i) = c_i(q_i)^2$ in each plant $i = \{1,2\}$, where $c_i > 0$. That is the total cost of plant 1 is $TC_1(q_1) = c_1(q_1)^2$, while that of plant 2 is $TC_2(q_2) = c_2(q_2)^2$, where parameter $c_1 \neq c_2$. The monopolist faces an inverse demand function $p(q_1, q_2) = 1 - (q_1 + q_2)$.

a) Set up the monopolist profit-maximization problem where it seeks to maximize its joint profit from both plants $\pi_1 + \pi_2$.

   Answer: Monopolist's total profit from both plants is
   $$\pi_1 + \pi_2 = [1 - (q_1 + q_2)]q_1 - c_1(q_1)^2 +$$
   $$[1 - (q_1 + q_2)]q_2 - c_2(q_2)^2$$
   where the first line represents the profits from plant 1 (total revenue minus total costs) and the second line measures the profits from plant 2 (total revenue minus total costs).

b) Differentiate with respect to the output level in each plant, q_1 and q_2, and find the profit-maximizing output in each plant, q_1^* and q_2^*.

   Answer: Differentiating with respect to q_1, we obtain
   $$1 - 2(q_1 + q_2) = 2c_1 q_1.$$
   And differentiating with respect to q_2, we find
   $$1 - 2(q_1 + q_2) = 2c_2 q_2$$
   Inserting one into the other, we obtain
   $$q_1^* = \frac{c_2}{2(c_1 + c_2 + c_1 c_2)} \quad \text{and} \quad q_2^* = \frac{c_1}{2(c_1 + c_2 + c_1 c_2)}$$

c) Find the marginal cost in each plant, $MC_i(q_i)$, and evaluate it at the profit-maximizing output on that plant found in part (b), that is, $MC_i(q_i^*)$ for every plant i. Check that the marginal cost coincides across plants. Intuitively, this indicates that the monopolist does not have further incentives to move production from one plant to another.

   Answer: Given $MC_i = 2c_i q_i$,we can evaluate $MC_i(q_i^*)$ for every plant i , that is,
   $$MC_1(q_1^*) = 2c_1 * \frac{c_2}{2(c_1 + c_2 + c_1 c_2)} = \frac{c_1 c_2}{(c_1 + c_2 + c_1 c_2)}$$
   And,
   $$MC_2(q_2^*) = 2c_2 * \frac{c_1}{2(c_1 + c_2 + c_1 c_2)} = \frac{c_1 c_2}{(c_1 + c_2 + c_1 c_2)}$$
   Obviously, $MC_1(q_1^*) = MC_2(q_2^*)$, indicating that the monopolist does not have further incentives to move production from one plant to another.

Exercise #7 - Price discrimination with linear costs

Consider a monopoly facing inverse demand function $p(q) = 100 - q$, and total cost $TC(q) = 4q$.

a) *No price discrimination*. Assume that price discrimination is illegal. What are the monopolist's optimal output, price and profits?

b) *First-degree price discrimination*. For the remainder of the exercise, consider now that the monopolist can practice price discrimination. In addition, assume that this firm has enough information to practice first-degree (perfect) price discrimination. What are the monopolist's optimal output, price and profits?

c) *Two block pricing*. Assume that the monopolist offers price discounts (i.e., two blocks of units, each sold at a different price per unit). What are the monopolist's optimal output, price and profits?

d) *Comparison*. Compare the monopolist's profit under each of the above pricing strategies, and show that the profits in part (b) are the highest, followed by those in (c), followed by those in part (c), and by those in part (a).

Answer:

a) With no price discrimination, the monopoly optimal price and output occur when $MR = MC$. In this setting,

$$MR = MC$$

$$100 - 2q = 4$$

Rearranging, we obtain $2q = 96$, and solving for q gives us

$$q = 48$$

Plugging this back into our demand will give us our optimal price.

$$p(q) = 100 - 48 = \$52$$

Finally, we plug these into our profit function. Thus,

$$\pi = pq - cq = (52)(48) - 4(48) = 2{,}496 - 192 = \$2{,}304.$$

b) With first degree price discrimination, the entire demand curve becomes the marginal revenue for the firm because they charge each customer their willingness to pay. Hence, we solve by setting our demand equal to marginal cost.

$$100 - q = 4$$

Solving for q , we obtain $q = 96$. (Because the firm charges each consumer a different price, we can't solve for p.)

Finally, our profit can be found by finding producer surplus. In this setting,

$$PS = \frac{1}{2}(100 - 4)(100 - 4) = \frac{1}{2}(96)(96) = \$4{,}608$$

c) First, we set up our first block. In this setting,

$$p_1 q_1 = (100 - q_1)q_1$$

Next, we define our second block

$$p_2(q_2 - q_1) = (100 - q_2)(q_2 - q_1)$$

Our total cost in this case is $MC * q_2 = cq_2$ because we are selling q_2 units in total. Setting up our profit gives us,

$$\pi = p_1 q_1 + p_2(q_2 - q_1) - 4q_2$$

$$\pi = (100 - q_1)q_1 + (100 - q_2)(q_2 - q_1) - 4q_2$$

Simplifying gives us

$$\pi = 100q_2 + q_1q_2 - q_1^2 - q_2^2 - 4q_2$$

Next we differentiate with respect to q_1 and q_2.

$$\frac{\partial\pi}{\partial q_1}: q_2 - 2q_1 = 0, \quad \text{which yields } q_1 = \frac{q_2}{2}, \text{ and}$$

$$\frac{\partial\pi}{\partial q_2}: 100 + q_1 - 2q_2 - 4 = 0, \quad \text{which simplifies to } q_2 = \frac{96+q_1}{2}$$

Plugging q_2 into q_1 gives us

$$q_1 = \frac{1}{2}\left(\frac{96 + q_1}{2}\right)$$

Rearranging, and solving for q_1, we obtain

$$q_1 = 24 * \frac{4}{3} = 32$$

We plug this back into our q_2 function above to solve for the optimal q_2. In this setting,

$$q_2 = \frac{96 + 32}{2} = 64$$

Next, we plug these into our price functions to find our optimal prices. First, we solve for p_1

$$p_1 = 100 - q_1 = 100 - 32 = 68$$

Solving for p_2

$$p_2 = 100 - 64 = 36$$

Finally, we plug these back into our profit function to solve our optimal profit.

$$\pi = p_1q_1 + p_2(q_2 - q_1) - 4q_2 = 68 * 32 + 36 * (64 - 32) - 4 * 64 = \$3{,}072.$$

d) Summarizing the following profits, we found

$$Part\ A: \$2{,}304$$

$$Part\ B: \$4{,}608$$

$$Part\ C: \$3{,}072$$

Thus, we can see that our profits are highest with first degree price discrimination (b), followed by two block pricing (c), and finally no price discrimination (a).

Exercise #8 - Third-degree price discrimination with convex costs

Consider a monopolist facing two groups of costumers, 1 and 2, with inverse demand functions $p_1(q) = a_1 - bq$ and $p_2(q) = a_2 - bq$, respectively, where $a_1 > a_2 > 0$ and $b > 0$. The monopolist has convex cost function $TC(q) = c(q)^2$ where $c > 0$.

a) Set up the monopolist's profit-maximization problem for each group of customers.

Answer: The monopolist sets $MR_i = MC$, where $i = 1,2$ represents each group of customers. We obtain,

$$\frac{\partial\,(p_1(q)*q)}{\partial q} = \frac{\partial(a_1 - bq)q}{\partial q} = a_1 - 2bq \text{ and } \frac{\partial\ TC(q)}{\partial q} = \frac{\partial c(q)^2}{\partial q} = 2cq$$

Then $MR_1 = MC$ leads to,

$$a_1 - 2bq = 2cq$$

And similarly for the second group,

$$\frac{\partial\,(p_2(q)*q)}{\partial q} = \frac{\partial(a_2 - bq)q}{\partial q} = a_2 - 2bq \text{ and } \frac{\partial\ TC(q)}{\partial q} = \frac{\partial c(q)^2}{\partial q} = 2cq$$

Then $MR_{\ 2} = \ MC$ leads to,

$$a_2 - 2bq = 2cq$$

b) Find its profit-maximizing output and price for each group of customers.

Answer: Given the results from part a, we immediately obtain,

$$a_i - 2bq = 2cq$$
$$2bq + 2cq = a_i$$
$$q = \frac{a_i}{2b + 2c}$$

Thus, our demand for group 1 is

$$q_1 = \frac{a_1}{2(b + c)}$$

Similarly, for group 2

$$q_2 = \frac{a_2}{2(b + c)}$$

First, we plug in the demand for q_1 into our p_1 function to find our optimal price for group 1

$$p_1 = a_1 - b\left(\frac{a_1}{2(b + c)}\right) = \frac{a_1(b + 2c)}{2(b + c)}$$

Next, we plug in our second demand to find the optimal price for group 2

$$p_2 = a_2 - b\left(\frac{a_2}{2(b + c)}\right) = \frac{a_2(b + 2c)}{2(b + c)}$$

c) Assume that $a_1 = ta_2$, where $t > 1$. Evaluate your above results using $a_1 = ta_2$, and determine how the output difference across groups of customers, and the price difference, are affected by a larger value of t.

Answer: Using $a_1 = ta_2$ and substituting into optimum price and quantity for group 1, we obtain,

$$p_1 = \frac{ta_2(b + 2c)}{2(b + c)}$$

And,

$$q_1 = \frac{ta_2}{2(b + c)}$$

Therefore, the price difference between the two groups is

$$\Delta p = p_1 - p_2 = \frac{ta_2(b + 2c)}{2(b + c)} - \frac{a_2(b + 2c)}{2(b + c)} = \frac{a_2[(t - 1)(b + 2c)]}{2(b + c)}$$

And the output difference is

$$\Delta q = q_1 - q_2 = \frac{ta_2}{2(b + c)} - \frac{a_2}{2(b + c)} = \frac{a_2(t - 1)}{2(b + c)}$$

d) What would happen if $t = 1$, so the inverse demand functions coincide for both groups of customers?

Answer: Given the result above, if $t = 1$, then $\Delta p = \Delta q = 0$, implying the optimum prices and quantities are identical. This comes as no surprise since in this case the inverse demand functions for both groups coincide.