EconS 301 – Intermediate Microeconomics with Calculus

Midterm Exam #1 – *Answer key*

Exercise #1. Consider a consumer with the following Cobb-Douglas utility function for two goods, 1 and 2,

$$u(x_1, x_2) = x_1^{1/2} x_2^{1/3}$$

Assume that the consumer faces a price of \$1 for good 2, and a total income of \$*I*. The price of good 1 is left unrestricted as p_1.

a) Find the marginal utility of good 1 and of good 2.

$$MU_{x_1} = \frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}} = \frac{x_2^{\frac{2}{3}}}{3x_1^{\frac{2}{3}}}$$

$$MU_{x_2} = \frac{2}{3} x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}} = \frac{2x_1^{\frac{1}{3}}}{3x_2^{\frac{1}{3}}}$$

b) Given your results in part (a), does the "more is better" property hold for this consumer?

Yes, because both MU_{x_1} and MU_{x_2} are both strictly positive for any values of x_1 and x_2. This result reflects that, as we increase our consumption of goods x_1 and x_2, we increase our utility.

c) Use the marginal utilities found in part (a) to find the marginal rate of substitution, *MRS*, for this consumer.

$$MRS = \frac{MU_{x_1}}{MU_{x_2}} = \frac{\frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}}}{\frac{2}{3} x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}}} = \frac{x_2^{\frac{2}{3}}}{3x_1^{\frac{2}{3}}} \cdot \frac{3x_2^{\frac{1}{3}}}{2x_1^{\frac{1}{3}}}$$

Simplifying gives us

$$\frac{3x_2}{6x_1} = \frac{x_2}{2x_1}$$

d) Is the MRS found in part (c) increasing or decreasing in x_1? What does that tell us about the shape of indifference curves? Interpret.

The *MRS* is decreasing in part c because x_1 is in the denominator. As $x_1 \uparrow$, $MRS \downarrow$. The shape of the IC will be convex (bowed in towards the origin). That is, the IC becomes flatter as we move rightward towards more units of x_1.

e) Set up this consumer's utility maximization problem (UMP), and find the demand for good 1 and 2.
[*Hint*: Write the tangency condition, solve for x_2, and insert your result into the consumer's budget line. Solving for x_1, you will obtain the demand for good 1. Recall that the expression you find should be a function of the price of good 1, p_1.]

First, we set up our tangency condition $MRS = \frac{p_1}{p_2}$, which in this context is

$$\frac{x_2}{2x_1} = \frac{p_1}{1}$$

Solving for x_2 gives us

$$x_2 = 2p_1x_1$$

Plugging this into our budget constraint,

$$p_1x_1 + p_2x_2 = I$$
$$p_1x_2 + 1(2p_1x_1) = I$$
$$3p_1x_2 = I$$

which, solving for x_1 gives us the demand for good 1:

$$x_1 = \frac{I}{3p_1}$$

Plugging this back into the result we found from the tangency condition, $x_2 = 2p_1x_1$, we obtain

$$x_2 = 2p_1\left(\frac{I}{3p_1}\right) = \frac{2I}{3}$$

f) Is good 1 normal or inferior? What about good 2?

Both goods are normal because, as income increases, its demand increases (note that both demands have the income *I* on the numerator).

g) For the remainder of the exercise, you can assume an income of I=\$100, and that the price of good 1 decreases from $p_1 = \$4$ to $p_1 = \$2$. Find the increase in consumer surplus, CS, that this consumer enjoys from the price decrease.
[*Hint*: You will need to use a simple integral we saw in class, since using the area of a triangle won't give you a precise answer.]

Since the demand for good 1 is $x_1 = \frac{I}{3p_1}$, and income is \$100, the demand becomes $x_1 = \frac{100}{3p_1}$. Therefore, the gain in consumer surplus from the price decrease is measured by the integral of $\frac{100}{3p_1}$ between prices $p_1 = \$4$ to $p_1 = \$2$, as follows

$$CS = \int_2^4 \frac{100}{3p_1} dp_1 = \frac{100}{3} \int_2^4 \frac{1}{p_1} dp_1$$

where we moved the constant, $\frac{100}{3}$, outside the integral operator. Solving the integral gives us

$$CS = \frac{100}{3}(\ln 4 - \ln 2) = 23.105$$

h) Consider again the price decrease described in part (g). Find the compensating variation (CV).

To find our CV, we must subtract the income needed to buy the initial bundle A from the income needed to buy the decomposition bundle B. That is,

$$CV = I_A - I_B$$

Bundle A. First, we need to find our bundle A. To find our optimal bundle A, we must set up our tangency conditions

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{p_1}{p_2}$$

$$\frac{x_2}{2x_1} = \frac{4}{1}$$

Solving for x_2 gives us,

$$x_2 = 8x_1$$

Plugging this result into our budget constraint gives us

$$p_1x_1 + p_2x_2 = I$$

$$4x_1 + 1(8x_1) = 100$$

Solving for x_1 gives us,

$$x_1 = \frac{100}{12} = 8.33$$

We can then plug this into our result from the tangency condition, $x_2 = 8x_1$, as follows

$$x_2 = 8x_1 = 8(8.33) = 66.67$$

Thus our optimal bundle A, at initial price $p_1 = \$4$, is

$$A: (8.33, 66.67)$$

Bundle B. Next, we must find our decomposition bundle B. Recall that this bundle must satisfy two properties: (1) it must yield the same utility as bundle A; and (2) it must achieve a tangency between indifference curve and budget line at the new price ratio.

We start by finding our utility for bundle A. Plugging our values from bundle A into our utility function gives us

$$U_A = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}} = (8.33)^{\frac{1}{3}} (66.67)^{\frac{2}{3}} = 33.33$$

Hence, bundle B must give the consumer a utility of

$$x_1^{\frac{1}{3}} x_2^{\frac{2}{3}} = 33.33$$

Next, we set up our tangency conditions at the new price $p_1 = 2$, which gives us

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{p_1}{p_2}$$

$$\frac{x_2}{2x_1} = \frac{2}{1}$$

Solving for x_2 we obtain

$$x_2 = 4x_1$$

Plugging this into our utility function gives us

$$33.33 = (x_1)^{\frac{1}{3}} (4x_1)^{\frac{2}{3}}$$

Solving for x_1 yields $x_1 = 13.23$. We plug this back into our value for x_2 which gives us

$$x_2 = 4x_1 = 4 * 13.23 = 52.91$$

Thus our optimal consumption for bundle B is

$$B: (13.23, 52.91)$$

Next, we must find the cost of buying bundle bundle B at the new price

$$I_B = 2(13.23) + 1(52.91) = 79.37$$

As a consequence, the CV becomes

$$CV = I_A - I_B = 100 - 79.37 = 20.63$$

i) Consider again the same price decrease described in part (g). Find the equivalent variation (EV).

The expression of the EV is

$$EV = I_E - I_A$$

Bundle E. Hence, to find the EV, we must first find bundle E. We know that bundle E must satisfy two properties: (1) it must provide the same utility as bundle C; and (2) it must achieve a tangency between the indifference curve and the budget line at the initial prices.

We must first find bundle C (optimal consumption bundle at the new prices). To find this bundle, we must set up our tangency condition at our new price $p_1 = \$2$, that is

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{p_1}{p_2}$$

$$\frac{x_2}{2x_1} = \frac{2}{1}$$

Solving for x_2 gives us,

$$x_2 = 4x_1$$

Plugging this into our budget constraint, we obtain

$$p_1x_1 + p_2x_2 = I$$

$$4x_1 + 1(4x_1) = 100$$

Solving for x_1

$$x_1 = \frac{100}{8} = 16.67$$

We can then plug this into our result from the tangency condition $x_2 = 4x_1$, finding

$$x_2 = 4x_1 = 4(16.67) = 66.67$$

Thus our optimal bundle C at $p_1 = \$2$ is

$$C: (16.67,66.67)$$

Since bundle E has the same utility as bundle C, we next find the utility for bundle C

$$U_c = x_1^{\frac{1}{3}}x_2^{\frac{2}{3}} = (16.67)^{\frac{1}{3}}(66.67)^{\frac{2}{3}} = 42$$

Hence, bundle E must satisfy

$$x_1^{\frac{1}{3}}x_2^{\frac{2}{3}} = 42.$$

In addition, at bundle E we must have a tangency between the indifference curve and the budget line at the new prices. From our above discussion, we know that such a tangency condition yields $x_2 = 8x_1$. Inserting this result into the above utility level of bundle E, we obtain that

$$x_1^{\frac{1}{3}}x_2^{\frac{2}{3}} = (x_1)^{\frac{1}{3}}(8x_1)^{\frac{2}{3}} = 42$$

Solving for x_1 gives us

$$x_1 = \frac{42}{8^{\frac{2}{3}}} = 10.5$$

We can then plug this into our tangency condition $x_2 = 8x_1$, we find that

$$x_2 = 8x_1 = 8 * 10.5 = 84$$

Hence, bundle E is

$$E: (10.5, 84)$$

Since we found the consumption of bundle E, we now find the cost of buying bundle E at our initial prices

$$I_E = \; 4(10.5) + 1(84) = 126$$

To find our EV, we subtract the cost of bundle E from the cost of bundle A.

$$EV = I_E - I_A = 126 - 100 = 26$$

Exercise #2. Consider again the setting in Exercise #1, where income is I=\$100, and the price of good 1 decreases from $p_1 = \$4$ to $p_1 = \$2$. Also recall that the price of good 2 is fixed at $p_2 = \$1$.

a) Find the optimal consumption bundle at the *initial* price $p_1 = \$4$.

To find our optimal bundle, we must set up our tangency condition

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{p_1}{p_2}$$

$$\frac{x_2}{2x_1} = \frac{4}{1}$$

Solving for x_2 gives us,

$$x_2 = 8x_1$$

Plugging this result into our budget constraint gives us

$$p_1x_1 + p_2x_2 = I$$

$$4x_1 + 1(8x_1) = 100$$

Solving for x_1 we obtain

$$x_1 = \frac{100}{12} = 8.33$$

We can then plug this into the equation we obtained from the tangency condition, to find

$$x_2 = 8x_1 = 8(8.33) = 66.67$$

Thus our optimal bundle at the initial price $p_1 = \$4$ is

$$A: (8.33, 66.67)$$

b) Find the optimal consumption bundle at the *final* price $p_1 = \$2$.

To find our optimal bundle, we must set up our tangency conditions at our new price $p_1 = \; 2$

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{p_1}{p_2}$$

$$\frac{x_2}{2x_1} = \frac{2}{1}$$

Solving for x_2 gives us,

$$x_2 = 4x_1$$

Plugging this into our budget constraint, we obtain

$$p_1x_1 + p_2x_2 = I$$

$$4x_1 + 1(4x_1) = 100$$

Solving for x_1 gives us

$$x_1 = \frac{100}{8} = 16.67$$

We can then plug this into our result from the tangency condition to obtain

$$x_2 = 4x_1 = 4(16.67) = 66.67$$

Thus our optimal bundle at the final price $p_1 = \$2$ is

$$C: (16.67{,}66.67)$$

c) Find the increase in the consumption of good 1 due to the price decrease in p_1. This is the total effect.

To find the total effect, we subtract our consumption of x_1 at the initial price $p_1 = 4$ from our consumption of x_1 at the new price $p_1 = 2$.

$$TE = 16.67 - 8.33 = 8.33$$

That is, the consumer increases his consumption of the good that became relatively cheaper (good 1) by 8.33 units.

d) We next seek to break down the total effect you found in part (c) into the substitution and income effects. In order to do that, let us start by finding the decomposition bundle. Label it bundle *B*.
[*Hint*: Recall that the decomposition bundle must satisfy two conditions: (1) it must generate the same utility level as the initial bundle *A*; and (2) we must have that the slope of the consumer's indifference curve, *MRS*, coincides with the new price ratio.]

We start by finding our utility for bundle A. Plugging bundle A into our utility function, we obtain

$$U_A = x_1^{\frac{1}{3}}x_2^{\frac{2}{3}} = (8.33)^{\frac{1}{3}}(66.67)^{\frac{2}{3}} = 33.33$$

implying that bundle B must generate the same utility level. That is,

$$U_B = x_1^{\frac{1}{3}}x_2^{\frac{2}{3}} = 33.33$$

From part (b) we know from our tangency condition at the newe prices is $x_2 = 4x_1$. Plugging that into our utility function gives us

$$33.33 = (x_1)^{\frac{1}{3}}(4x_1)^{\frac{2}{3}}$$

Solving for x_1 we obtain

$$x_1 = 13.23$$

We plug this back into our tangency condition, to find

$$x_2 = 4x_1 = 4 * 13.23 = 52.91$$

Thus the decomposition bundle B is

$$B: (13.23, 52.91)$$

e) Write the amount of good 1 that this individual consumes on bundles *A*, *B* and *C*. What is the increase in consumption of good 1 due to the substitution effect? What is due to the income effect?

From above, we found the following three bundles

$$A: (8.33, 66.67)$$
$$B: (13.23, 52.91)$$
$$C: (16.67, 66.67)$$

Thus our consumption of x_1 on bundles *A*, *B*, and *C* are 8.33, 13.23, and 16.67 respectively.

To find the substitution effect, we subtract the consumption of bundle B from the consumption of bundle A

$$SE = B - A = 13.23 - 8.33 = 4.9$$

To find the income effect, we subtract the consumption of bundle C from the consumption of bundle B.

$$IE = C - B = 16.67 - 13.23 = 3.44$$

Exercise #3. In class we presented a compact expression of the Slutsky equation using elasticities. Use this equation to answer the following two questions. (Writing the Slutsky equation and a short verbal explanation suffices.)

a) The income effect is negligible (close to zero) in goods such as garlic.

Our Slutsky equation is given as

$$\varepsilon = \varepsilon^* - \varepsilon_{Q,I}\theta$$
$$TE = SE - IE$$

When $\theta = 0$, there is no income effect, so the total effect is equal to the substitution effect. We usually see this with and goods that represent a negligible budget share in a consumer's purchases.

b) The income effect is large in goods such as housing.

Again, our Slutsky equation is

$$\varepsilon = \varepsilon^* - \varepsilon_{Q,I}\theta$$
$$TE = SE - IE$$

If θ is very large, the income effect will be very large. We would need a large wealth compensation to reach the same utility level as before the price change.