Domination Through Mixed Strategies
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When using IDSDS as a solution concept, many students have difficulty determining if a strategy can be dominated by a mixed strategy. The purpose of this exercise is to guide students through the process of domination through mixed strategies as well as to offer a few practice exercises.

First of all, when using IDSDS, we have shown that for a strategy to be strictly dominated, the following relationship must hold:

$$v_i(s_i, s_{-i}) > v_i(s_i', s_{-i}) \; \forall s_{-i} \in S_{-i}$$

meaning that no matter what strategies the other players use, the strategy s_i for player i gives a strictly higher payoff than playing any other strategy s_i'. Moving forward with domination through mixed strategies we will generalize this requirement slightly. Our relationship now becomes:

$$v_i(\sigma_i, s_{-i}) > v_i(s_i', s_{-i}) \; \forall s_{-i} \in S_{-i}$$

where σ_i now represents some mixed strategy being used by player i (remember that all pure strategies are also mixed strategies, just with degenerate probabilities). What this relationship boils down to is that if we can find only *one* mixed strategy for player i that strictly dominates one of player i's pure strategies, we can delete that pure strategy. Let's look at a sample inspired by *Game Theory: An Introduction* by Steven Tadelis (2013, pp. 115). Consider the following game in figure 1:

		Player 2		
		L	*C*	*R*
	U	5, 1	1, 4	1, 0
Player 1	*M*	3, 2	0, 0	3, 5
	D	4, 2	4, 4	0, 3

Figure 1

First of all, there are no strictly dominance in pure strategies. But could it be possible that one strategy is dominated through a mixed strategy? The first step for determining this is to figure out the best responses for both players for this game. This is shown in figure 2 below

		Player 2		
		L	*C*	*R*
	U	$\underline{5}$, 1	1, $\underline{4}$	1, 0
Player 1	*M*	3, 2	0, 0	$\underline{3}$, $\underline{5}$
	D	4, 2	$\underline{4}$, 4	0, $\underline{3}$

Figure 2

Where, for completeness, we define $BR_1(L) = U$, $BR_1(C) = D$, $BR_1(R) = M$, and $BR_2(U) = C$, $BR_2(M) = R$, $BR_2(D) = R$. Looking at the best response functions, we can see that player 2 never chooses strategy L regardless of the strategy that player 1 chooses. This makes strategy L a candidate for being dominated by a mixed strategy (Note that it would also be a candidate if Player 2 were indifferent between L and another strategy, as in the sample shown by Tadelis (2013)). The only possible mixed strategy we could use to strictly dominate L would be a mix between C and R. Let q denote the probability with which player 2 chooses strategy C, and $1 - q$ denote the probability with which player 2 chooses strategy R. We need to find a value for q such that the Player 2 will always prefer to play the mixed strategy between C

and R regardless of what Player 1 chooses. This can be shown in the following three expressions:

$$\text{Player 1 chooses } U \quad : \quad q * \overbrace{4}^{C} + (1-q) * \overbrace{0}^{R} > \overbrace{1}^{L} \implies q > \frac{1}{4}$$

$$\text{Player 1 chooses } M \quad : \quad q * 0 + (1-q) * 5 > 2 \implies q < \frac{3}{5}$$

$$\text{Player 1 chooses } D \quad : \quad q * 4 + (1-q) * 3 > 2 \implies q > -1$$

Hence, for all three of these expressions to hold true, we know that $\frac{1}{4} < q < \frac{3}{5}$. Any value for q that is between $\frac{1}{4}$ and $\frac{3}{5}$ will provide a mixed strategy between C and R that will strictly dominate the pure strategy L. For example, let's consider $q = \frac{1}{2}$. This would cause our expected payoffs from playing the mixed strategy to be

$$\text{Player 1 chooses } U \quad : \quad \frac{1}{2} * 4 + \left(1 - \frac{1}{2}\right) * 0 = 2 > 1$$

$$\text{Player 1 chooses } M \quad : \quad \frac{1}{2} * 0 + \left(1 - \frac{1}{2}\right) * 5 = 2.5 > 2$$

$$\text{Player 1 chooses } D \quad : \quad \frac{1}{2} * 4 + \left(1 - \frac{1}{2}\right) * 3 = 3.5 > 1$$

We can also show this in our game, as shown in figure 3:

		Player 2	
		L	*½ C + ½ R*
	U	5, 1	1, 2
Player 1	*M*	3, 2	1.5, 2.5
	D	4, 2	2, 3.5

Figure 3

where it is clear that the mixed strategy of $(\frac{1}{2}C\frac{1}{2}R)$ strictly dominates the pure strategy L. Therefore, we can delete strategy L from our game tree and are left with the following reduced normal form game as seen in figure 4:

		Player 2	
		C	*R*
	U	1, 4	1, 0
Player 1	*M*	0, 0	3, 5
	D	4, 4	0, 3

Figure 4

This game can be further reduced by domination through mixed strategies, but that will be saved for an exercise. Let's look at one more example in figure 5. This example is directly from one of my own homeworks in the PhD level game theory course (But don't worry, it's easy!):

		Player 2	
		d	*e*
	A	2, <u>0</u>	2, <u>0</u>
Player 1	*B*	<u>3</u>, 0	0, <u>1</u>
	C	0, <u>1</u>	<u>3</u>, 0

Figure 5: Devocalized Spindle Cell

In this game, Player 1 never chooses strategy A as a best response to either of Player 2's strategies, and therefore it is a candidate for domination by a mixed strategy of B and C (As a note, any non-square game will always have at least 1 candidate). Let p be the probability that Player 1 plays B, and $1-p$ be the probability that Player 1 plays C. As before, for this mixed strategy to strictly dominate strategy A, it must be superior for at least one value of q regardless of what Player 2 chooses. Our equilibrium conditions are therefore:

$$\text{Player 2 chooses } d \quad : \quad p * \overbrace{3}^{B} + (1-p) * \overbrace{0}^{C} > \overbrace{2}^{A} \implies p > \frac{2}{3}$$

$$\text{Player 2 chooses } e \quad : \quad p * 0 + (1-p) * 3 > 2 \implies p < \frac{1}{3}$$

Well, that's not good. There does not exist a value of p that is both larger than $\frac{2}{3}$ and less than $\frac{1}{3}$ at the same time. Hence, there does not exist a mixed strategy between B and C that strictly dominates strategy A and we can't delete it from this game. In fact, the mixed strategy Nash Equilibrium of this game has Player 1 playing the pure strategy A as his solution (but we're not worried about that here)!

Here are a couple of practice exercises for consideration and practice (Good midterm practice!)

1. Finish the first example shown (Starting in figure 4). Show which strategies survive IDSDS and domination through mixed strategies.

2. Consider the game from Review Session 4 (Which you all should be coming to) in Figure 6. Find *two different* mixed strategies (as in a mix between a and c, a and d, or c and d) that dominate strategy b for 001. List all possible values of p for which these mixed strategies will dominate strategy b. Explain why all three different mixed strategies do not dominate strategy b.

		002: x	002: y
001	a	12, 0	0, 6
001	b	11, 1	1, 5
001	c	10, 2	4, 2
001	d	9, 3	6, 0

Figure 6. Avalanche Game