Microeconomic Theory I
Assignment #9 - *Answer key*

1. [**Private contributions to a public good**] Consider an economy with 2 consumers, Alessandro and Beatrice, $i = \{A, B\}$, one private good x, and one public good G. Let each consumer have an income of M. For simplicity, let the prices of both the public and private good to be 1. In addition, the utility functions of consumer A and B are:

$$\begin{aligned} U^A &= \log(x^A) + \log(G), \quad \text{for individual } A\text{, and} \\ U^B &= \log(x^B) + \log(G), \quad \text{for individual } B \end{aligned}$$

Assume that the public good G is only provided by the contributions of these two individuals, that is, $G = g^A + g^B$.

(a) Find Alessandro's best response function. Depict it in a figure with his contribution, g^A, on the vertical axis and Beatrice's contribution, g^B, on the horizontal axis.

- The utility maximization problem of Alessandro is that of selecting his consumption of private good, x, and his contribution to the public good, g^A, to solve

$$\max_{x, g^A} \log x^A + \log G$$

$$\text{subject to } \; x^A + g^A = M \; \text{ and } g^A + g^B = G$$

Taking into account that $x^A = M - g^A$, the above problem can be more compactly expressed as a program with a single choice variable,

$$\max_{g^A} \log(M - g^A) + \log(g^A + g^B)$$

Taking first order condition with respect to g^A yields

$$-\frac{1}{M - g^A} + \frac{1}{g^A + g^B} = 0$$

and solving for g^A we obtain:

$$g^A(g^B) = \frac{M}{2} - \frac{g^B}{2}$$

which represents individual A's best response function (see figure 1).

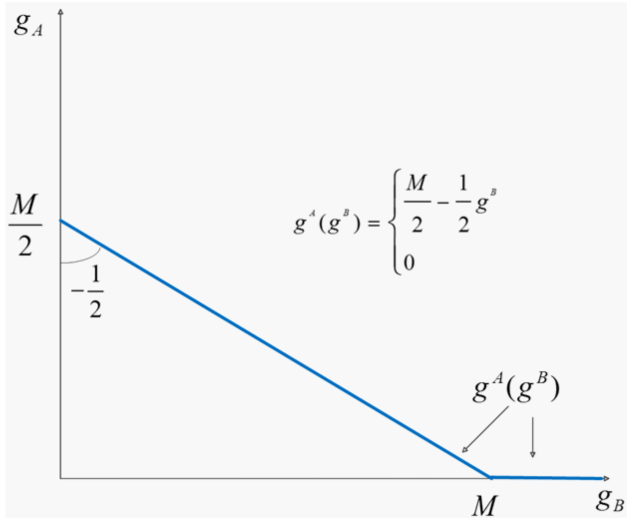

Figure 1. Alessandro's best response function.

- Intuitively, when Beatrice does not contribute to the public good, $g^B = 0$, Alessandro contributes $g^A = \frac{M}{2}$, but as Beatrice increases her contribution (rightward movements in figure 7.8), Alessandro responds by decreasing his own donation. In the extreme, when Beatrice donates all her wealth to the public good, i.e., $g^B = M$, Alessandro refrains from contributing, $g^A = 0$ for all $g^B \geqslant M$, as represented in the segment of his best response function, $g^A(g^B)$, that overlaps the horizontal axis in the right-hand side of figure 7.8.

(b) Identify Beatrice's best response function. Depict it in a figure with her contribution, g^B, on the horizontal axis and Alessandro's contribution, g^A, on the vertical axis.

- Similarly as for Alessandro, the utility maximization decision of Beatrice is that of selecting a contribution to the public good, g^B, that solves

$$\max_{g^B} \log(M - g^B) + \log(g^A + g^B)$$

with first order condition

$$-\frac{1}{M - g^B} + \frac{1}{g^A + g^B} = 0$$

solving for g^B yields

$$g^B(g^A) = \frac{M}{2} - \frac{g^A}{2}$$

which represents Beatrice's best response function (see figure 2). Note that we use the same axes as in the best response function of Alessandro, so we can afterwards superimpose both best response functions, $g^A(g^B)$ and $g^B(g^A)$, in the same figure to find their crossing point.

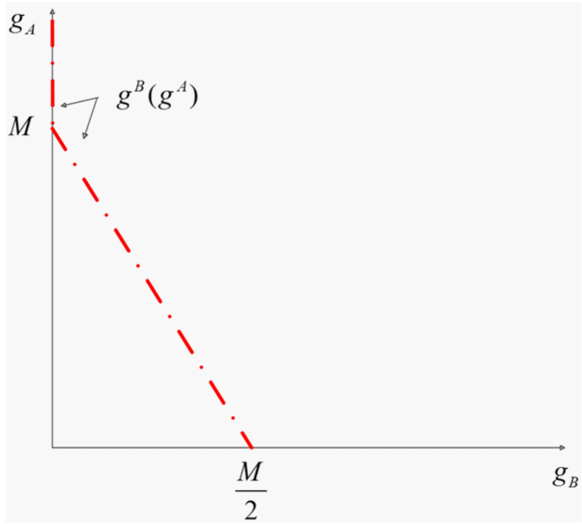

Figure 2. Beatrice's best response function.

(c) *Unregulated equilibrium.* Find the equilibrium contributions to the public good by Alessandro and Beatrice, that is, the Nash equilibrium of this public good game.

- Plugging Beatrice's best response function into Alessandro's best response function,

$$g^A = \frac{M}{2} - \frac{\left(\frac{M}{2} - \frac{g^A}{2}\right)}{2}$$

and solving for g^A, we obtain

$$g^A = \frac{M}{3}$$

which identifies the crossing point between both individuals' best response function, as depicted in figure 3. (A similar equilibrium contribution arises for Beatrice, $g^B = \frac{M}{3}$.)

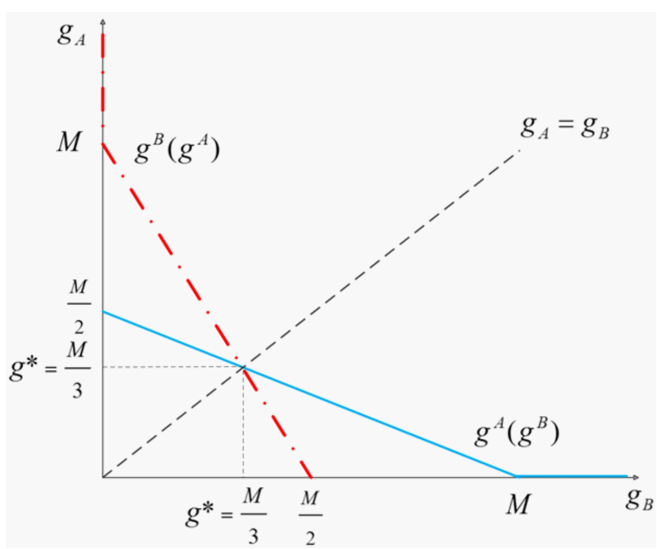

Figure 3. Equilibrium contributions to the public good.

Thus, the aggregate contribution to the public good in the Nash equilibrium is

$$g^A + g^B = \hat{G} = \frac{2M}{3}.$$

(d) *Social optimum.* Find the efficient (socially optimal) contribution to the public good by Alessandro and Beatrice.

- Recall that the utilitarian social welfare is $W = U^A + U^B$. The social planner must therefore choose individual contributions g^A and g^B to solve

$$\max_{g^A, g^B} \log\left(M - g^A\right) + \log(g^A + g^B) + \log\left(M - g^B\right) + \log(g^A + g^B)$$

Since individuals are symmetric, their optimal contributions must coincide, i.e., $g^A = g^B = g$, implying that we can simplify the above problem to

$$\max_{g} \ \log\left(M - g\right) + \log(g + g) + \log\left(M - g\right) + \log(g + g)$$

or

$$\max_{G} 2\log\left(M - g\right) + 2\log(2g)$$

and since $g + g = G$, we can further simplify the above problem to

$$\max_{G} \ 2\log\left(M - \frac{G}{2}\right) + 2\log(G)$$

Taking first order condition with respect to G, we obtain

$$-\frac{1}{2}\frac{2}{M - \frac{G}{2}} + \frac{2}{G} = 0.$$

Solving for G we find that the optimal aggregate contribution is $\widetilde{G} = M$. Hence, the sum of both individuals' contributions must add up to M. In a symmetric outcome this implies that each individual contributes half of this socially optimal level, that is

$$\widetilde{g}^A = \widetilde{g}^B = \frac{M}{2}.$$

- *Comparison.* Comparing $\widetilde{G} = M$ with $\hat{G} = \frac{2M}{3}$ shows that total provision at the Nash equilibrium, where every donor independently selects his own contribution, is below the socially optimal level, i.e., $\hat{G} < \widetilde{G}$.

(e) Use a figure to contrast the Pareto efficient level of private provision and the Nash equilibrium level of provision.

- Figure 4 compares individual contributions in the Nash equilibrium, $\hat{g} = \frac{M}{3}$, and socially optimal (Pareto efficient) contributions, $\frac{M}{2}$, which lie on the middle of the set of allocations satisfying $g^A + g^B = M$.

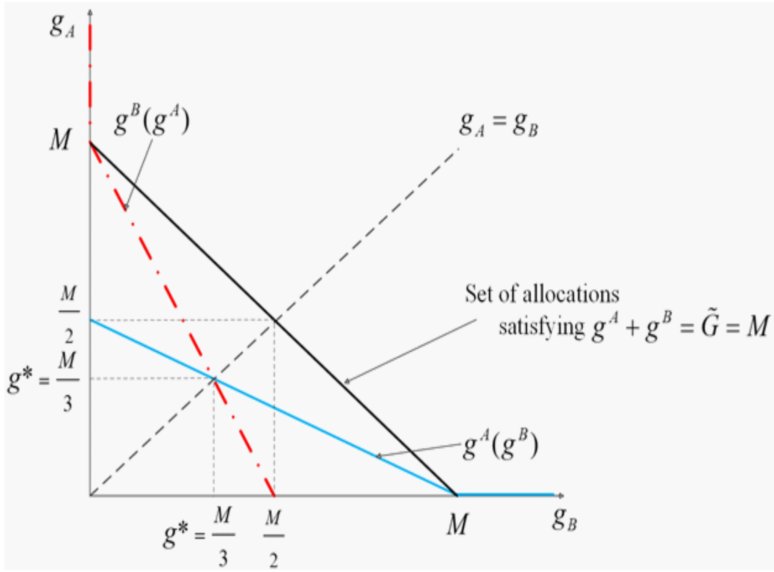

Figure 4. Equilibrium and socially optimal donations.

2. **[Production and Externalities]** According to some residents, a firm's production of paper at Lewiston, Idaho, generates a smelly gas as an unpleasant side product. Let $c(y, m; \mathbf{w})$ denote the (minimum) input cost of producing y tons of paper and m cubic meters of gas, where input prices are given by the vector $\mathbf{w} >> \mathbf{0}$. Let $p > 0$ denote the market price of paper. Assume that the cost function satisfies $\frac{\partial c}{\partial y} > 0$ and $\frac{\partial c}{\partial m} < 0$, and that $c(y, m; \mathbf{w})$ is strictly convex in y and m. Let stars * denote solutions and assume throughout that the firm produces positive amounts of paper $y^* > 0$.

   (a) Show that the cost function $c(y, m; \mathbf{w})$ is concave in input prices, $\mathbf{w}$.

   - Fix two input price vectors $\mathbf{w}$ and $\mathbf{w}'$ and consider their convex combination $\mathbf{w}'' = \alpha\mathbf{w} + (1-\alpha)\mathbf{w}'$, for any $\alpha \in (0, 1)$. Let $\mathbf{x}$ (respectively, $\mathbf{x}'$ and $\mathbf{x}''$) be the minimum cost bundle for input prices $\mathbf{w}$ (respectively, $\mathbf{w}'$ and $\mathbf{w}''$). By cost minimization we have

$$\begin{aligned} c(y, m; \mathbf{w}'') &= \alpha\mathbf{w}\mathbf{x}'' + (1-\alpha)\mathbf{w}'\mathbf{x}'' \\ &\geq \alpha\mathbf{w}\mathbf{x} + (1-\boldsymbol{\alpha})\mathbf{w}'\mathbf{x}' \\ &= \alpha c(y, m; \mathbf{w}) + (1-\alpha)c(y, m; \mathbf{w}') \end{aligned}$$

   So $c(y, m; \mathbf{w})$ is concave in input prices $\mathbf{w}$.

   (b) *Setting a quota.* Suppose that the government imposes a ceiling on gas emissions such that $m \leq \overline{m}$ (a quota). Assuming that this constraint binds, write down the firm's profit maximization problem with respect to y, and find necessary and sufficient conditions for the firm's cost-minimizing production, y^*,

   - The profit maximization problem for the firm is that of selecting an output level y that solves

$$\max_{y} \; py - c(y, m; \mathbf{w})$$
$$\text{subject to } m \leqslant \overline{m}$$

- If the constraint binds, $m = \overline{m}$, then the first order condition with respect to output, y, is

$$p = \frac{\partial c(y^*, \overline{m}; \mathbf{w})}{\partial y}$$

that is, price equals marginal cost at the optimum. Note that the constraint will be binding, i.e., $m = \overline{m}$. Otherwise, the firm would be able to further increase output and its associated profits.

(c) *Comparative statics.* Under which condition on the cost function $c(y, m; \mathbf{w})$ can we guarantee that an increase in the ceiling on gas emissions, $\overline{m}$, produces a raise in the firm's cost-minimizing production, y^*, whereby $\frac{\partial y^*}{\partial \overline{m}} > 0$?

- Differentiating the above expression again with respect to $\overline{m}$, we obtain

$$0 = \frac{\partial^2 c(y^*, \overline{m}; \mathbf{w})}{\partial y^2} \frac{\partial y^*}{\partial \overline{m}} + \frac{\partial^2 c(y^*, \overline{m}; \mathbf{w})}{\partial m \partial y}$$

and rearranging we obtain the usual expression of the implicit function theorem,

$$\frac{\partial y^*}{\partial \overline{m}} = -\frac{\frac{\partial^2 c(y^*, \overline{m}; \mathbf{w})}{\partial m \partial y}}{\frac{\partial^2 c(y^*, \overline{m}; \mathbf{w})}{\partial y^2}}$$

- Since the cost function $c(\cdot)$ is strictly convex in output y, the denominator is positive. Hence, a necessary and sufficient condition for $\frac{\partial y^*}{\partial \overline{m}} > 0$ is that $\frac{\partial^2 c(y^*, \overline{m}; \mathbf{w})}{\partial m \partial y} < 0$, i.e., an increase in the pollution ceiling, $\overline{m}$, reduces the *marginal* cost of production. As long as this (relatively reasonable) condition holds, an increase in the pollution ceiling $\overline{m}$ would induce the firm to increase production, that is, $\frac{\partial y^*}{\partial \overline{m}} > 0$.

(d) *Emission fee.* Suppose now that the government abandons its emissions ceiling and replaces it with a tax $t > 0$ on gas emissions. Thus, the new cost of producing (y, m) is given by $c(y, m; \mathbf{w}) + tm$. Show that maximized profits are convex in t, and that the firm's choice of pollution decreases in the pollution tax, i.e., $\frac{\partial m^*}{\partial t} \leq 0$.

- The profit maximization problem for the firm can now be written as selecting its output level and pollution to solve

$$\max_{y,m} \; py - c(y, m; \mathbf{w}) - tm$$

- Suppose (y, m), (y', m') and (y'', m'') maximize profits for tax levels t, t' and t'', respectively, where $t'' = \alpha t + (1 - \alpha)t'$ for any $\alpha \in (0, 1)$. By profit maximization it follows that

$$\begin{aligned} \pi(p, \mathbf{w}, t) &= py - c(y, m; \mathbf{w}) - tm \geq py'' - c(y'', m''; \mathbf{w}) - tm'', \text{ and} \\ \pi(p, \mathbf{w}, t') &= py' - c(y', m'; \mathbf{w}) - t'm' \geq py'' - c(y'', m''; \mathbf{w}) - t'm'' \end{aligned}$$

Hence, the convex combination of profit functions $\pi(p, \mathbf{w}, t)$ and $\pi(p, \mathbf{w}, t')$ yields

$$\begin{aligned} & \alpha\pi(p, \mathbf{w}, t) + (1 - \alpha)\pi(p, \mathbf{w}, t') \\ \geq \; & py'' - c(y'', m''; \mathbf{w}) - [\alpha t + (1 - \alpha)t']m'' \\ = \; & py'' - c(y'', m''; \mathbf{w}) - t''m'' = \pi(p, \mathbf{w}, t'') \end{aligned}$$

Then the profit function $\pi(p, \mathbf{w}, t)$ is convex in the tax level t. Intuitively, the maximal profit that the firm can obtain from a convex combination of fees t and t', i.e., $\pi(p, \mathbf{w}, t'')$ where $t'' = \alpha t + (1-\alpha)t'$, is lower than the convex combination of profits when the firm faces either a fee t or t', i.e., $\pi(p, \mathbf{w}, t'')$ lies below $\alpha\pi(p, \mathbf{w}, t) + (1-\alpha)\pi(p, \mathbf{w}, t')$, as figure 5 depicts.

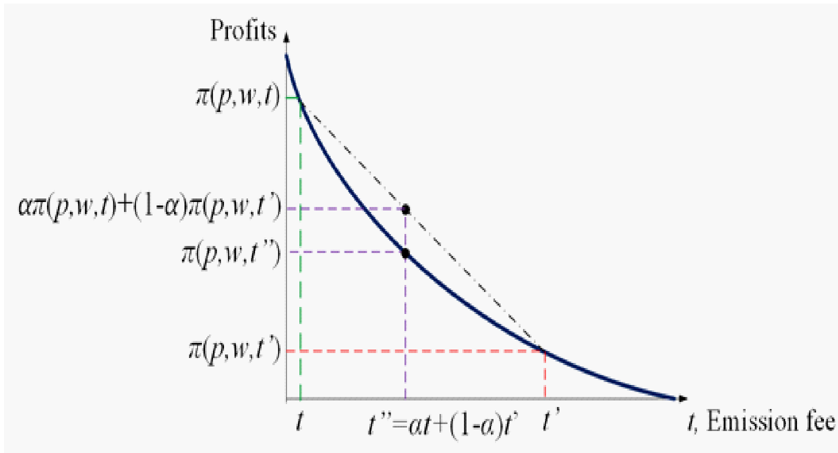

Figure 5. Convexity of profit function $\pi(p, w, t)$.

- Let $\mathbf{x}$ and $\mathbf{x}'$ be the input vectors for the profit-maximizing plans y and y' associated with taxes t and t', respectively. By profit maximization,

$$\pi(p, \mathbf{w}, t) = py - \mathbf{w}\mathbf{x} - tm \geq py' - \mathbf{w}\mathbf{x}' - tm'$$

and rearranging,

$$-p(y' - y) + \mathbf{w}(\mathbf{x}' - \mathbf{x}) + t(m' - m) \geq 0 \tag{1}$$

Similarly for tax level t',

$$\pi(p, \mathbf{w}, t') = py' - \mathbf{w}\mathbf{x}' - t'm' \geq py - \mathbf{w}\mathbf{x} - t'm$$

and rearranging,

$$p(y' - y) - \mathbf{w}(\mathbf{x}' - \mathbf{x}) - t'(m' - m) \geq 0 \tag{2}$$

Hence, by adding inequalities (1) and (2), we obtain

$$(t' - t)(m' - m) \leq 0$$

which means that the firm's choice of pollution level, m, decreases as the tax t increases, or in differential terms

$$\frac{\partial m^*}{\partial t} \leq 0.$$

3. [**Social planner prefering Cournot or Bertrand competition?**] Consider an industry with n symmetric firms, each facing a constant marginal cost $c > 0$ and inverse demand function $p(Q) = 1 - Q$, where $1 > c$. In addition, firms' production generates a linear environmental externality (damage) measured by $ED(Q) = d \times Q$.

(a) Assuming that firms compete a la Cournot, find their equilibrium individual and aggregate output, the equilibrium profits, the associated consumer surplus and overall social welfare.

- From similar exercises, we know that individual equilibrium output when firms compete a la Cournot is $q^C = \frac{1-c}{n+1}$ and thus aggregate equilibrium output is $Q^C = n\frac{1-c}{n+1}$. Equilibrium profits are, therefore,

$$\pi^C = \left(1 - n\frac{1-c}{n+1}\right)\frac{1-c}{n+1} - c\frac{1-c}{n+1} = \frac{(1-c)^2}{(n+1)^2}$$

while consumer surplus becomes

$$CS^C = \frac{\left(Q^C\right)^2}{2} = \frac{\left(n\frac{1-c}{n+1}\right)^2}{2} = \frac{n^2(1-c)^2}{2(n+1)^2}$$

implying that social welfare in equilibrium is

$$\begin{aligned} SW^C &= CS^C + n\pi^C - ED^C = \frac{n^2(1-c)^2}{2(n+1)^2} + n\frac{(1-c)^2}{(n+1)^2} - d \times n\frac{1-c}{n+1} \\ &= \frac{n(1-c)\left[(n+2)(1-c) - (n+1)2d\right]}{2(n+1)^2} \end{aligned}$$

(b) Assuming that firms compete a la Bertrand, find their equilibrium individual and aggregate output, the equilibrium profits, the associated consumer surplus and overall social welfare.

- When firms compete a la Bertrand, they practice marginal cost pricing, i.e., $p = c$. Hence, $p = 1 - Q^B = c$, which entails an aggregate equilibrium output of $Q^B = 1 - c$, yielding an individual equilibrium output of $q^B = \frac{1-c}{n}$ since firms are symmetric. Equilibrium output in this setting is zero, i.e.,

$$\pi^B = (1 - (1-c))\frac{1-c}{n} - c\frac{1-c}{n} = 0$$

whereas consumer surplus becomes

$$CS^B = \frac{\left(Q^B\right)^2}{2} = \frac{(1-c)^2}{2}$$

implying that social welfare in equilibrium is

$$\begin{aligned} SW^B &= CS^B + n\pi^B - ED^B = \frac{(1-c)^2}{2} + 0 - d \times (1-c) \\ &= \frac{(1-c)(1-c-2d)}{2} \end{aligned}$$

(c) Compare the social welfare arising when firms compete a la Cournot (found in part a) and a la Bertrand (found in part b). Under which conditions does the social planner prefer that firms compete a la Cournot? Interpret.

- *Without externalities.* Before explicitly comparing SW^C and SW^B note that, when the environmental externality is absent, i.e., $d = 0$, competition a la Bertrand produces a larger social welfare than competition a la Cournot. Intuitively, the larger production that arises under Bertrand generates an increase in consumer surplus that offsets the decrease in firms' profits, ultimately increasing social welfare.
- *With externalities.* When the environmental externality is present, however, the larger aggregate output under Bertrand entails welfare gains relative to Cournot, which originate from a larger value in the sum $CS^B + n\pi^B$ than in $CS^C + n\pi^C$, but it also produces a welfare loss, as such large production generates a larger environmental damage. Explicitly comparing SW^C and SW^B, we obtain

  $$SW^B - SW^C = \frac{(1-c)\left[1-c-2(n+1)d\right]}{2(n+1)^2}$$

  which is positive if and only if $1 - c > 2(n + 1)d$, or solving for parameter d,

  $$d < \frac{1-c}{2(n+1)}.$$

  In words, the Bertrand model produces a larger social welfare than the Cournot model when the environmental externality is sufficiently small, i.e., if $d < \frac{1-c}{2(n+1)}$. Note that this condition in parameter d embodies the case in which externalities are absent as a special case, $d = 0$, thus confirming the above intuition of Bertrand yielding a larger social welfare than Cournot. If, in contrast, the environmental externality is sufficiently damaging, i.e., $d \geq \frac{1-c}{2(n+1)}$, the larger production that arises under the Betrand model lowers social welfare below that emerging under Cournot.