Microeconomic Theory I
Assignment #8 - Answer key

1. [**Externalities and car accidents**] Consider an economy with two individuals $i = \{1, 2\}$ with the following quasi-linear utility function

$$u_i(s^i, q^i) = v^i(s^i) + \alpha w^i$$

where s^i denotes the speed at which individual i drives his car, w^i is his wealth, and $\alpha > 0$. The utility that individual i obtains from driving fast is $v^i(s^i)$, which is increasing but concave in speed, whereby $\frac{\partial v^i(s^i)}{\partial s^i} > 0$ and $\frac{\partial^2 v^i(s^i)}{(\partial s^i)^2} < 0$. Driving fast, however, increases the probability of suffering a car accident, represented by $\gamma(s^i, s^j)$. This probability is increasing both in the speed at which individual i drives, s^i, and the speed at which other individuals drive, s^j, where $j \neq i$. Hence, the speed of other individuals imposes a negative externality on driver i, since it increases his risk of suffering a car accident. If individual i suffers an accident, he bears a cost of $c^i > 0$, which intuitively embodies the cost of fixing his car, health-care expenses, etc.

   (a) *Unregulated equilibrium.* Set up individual i's expected utility maximization problem. Take first-order conditions with respect to s^i, and denote the (implicit) solution to this first-order condition as $\widehat{s}^i$.

   - With probability $\gamma(s^i, s^j)$, the individual suffers a car accident, and thus his utility is $v^i(s^i) + \alpha w^i - c^i$, and with probability $1 - \gamma(s^i, s^j)$ he does not suffer the accident, leaving his utility level at $v^i(s^i) + \alpha w^i$.
   - Hence, his expected utility is

$$\gamma(s^i, s^j)[v^i(s^i) + \alpha w^i - c^i] + (1 - \gamma(s^i, s^j))[v^i(s^i) - \alpha w^i],$$

     which reduces to $v^i(s^i) + \alpha w^i - \gamma(s^i, s^j)c^i$. Hence, every individual i maximizes his expected utility by choosing an speed level s^i that solves

$$\max_{s^i} \; v^i(s^i) + \alpha w^i - \gamma(s^i, s^j) \times c^i$$

     Taking first-order conditions with respect to s^i we obtain

$$\frac{\partial v^i(s^i)}{\partial s^i} - \frac{\partial \gamma}{\partial s^i} c^i = 0 \tag{1}$$

     Hence, driver i independently selects the speed, $\widehat{s}^i$, that solves $\frac{\partial v^i(s^i)}{\partial s^i} = \frac{\partial \gamma}{\partial s^i} c^i$.
   - Intuitively, driver i increases his speed s^i until the point where the additional utility from marginally increasing s^i, $\frac{\partial v^i(s^i)}{\partial s^i}$, coincides with its associated expected individual cost from speed, i.e., a higher probability of suffering a car accident times its associated cost, as measured by $\frac{\partial \gamma}{\partial s^i} c^i$.

- *Parametric example.* Consider, for instance, a utility from driving fast of $v(s^i) = \sqrt{s^i}$ (which is increasing and concave in s^i, as required), and that the probability of suffering a car accident is $\gamma(s^i, s^j) = \beta_i s^i + \beta_j s^j$, where $\beta_i > \beta_j$ (indicating that my own speed increases the probability that I suffer a car accident more than other drivers' speeds). First order condition (1) in this context becomes

$$\frac{1}{2\sqrt{s^i}} = \beta_i c^i,$$

and solving for s^i, we obtain an equilibrium speed of $\widehat{s}^i = \frac{1}{4(\beta_i c^i)^2}$ for every individual driver $i = \{1, 2\}$.

(b) *Social optimum.* Set up the social planner's expected welfare maximization problem. Take first-order conditions with respect to s^1 and s^2. Denote the (implicit) solution to this first-order condition as $\overline{s}^i$.

- The social planner solves the expected welfare maximization problem

$$\max_{s^1, s^2} \; v^1(s^1) + \alpha w^1 + v^2(s^2) + \alpha w^2 - \gamma(s^1, s^2) \times \left(c^1 + c^2\right)$$

Taking first-order conditions with respect to s^1, we obtain that $\bar{s}^1$ solves

$$\frac{\partial v^1(s^1)}{\partial s^1} = \frac{\partial \gamma}{\partial s^1}\left(c^1 + c^2\right) \tag{2}$$

and similarly with respect to s^2, we obtain that $\bar{s}^2$ solves

$$\frac{\partial v^2(s^2)}{\partial s^2} = \frac{\partial \gamma}{\partial s^2}\left(c^1 + c^2\right) \tag{3}$$

Intuitively, at the social optimum every driver i increases his speed s^i until the point where the additional utility from marginally increasing s^i coincides with its associated expected social cost from speed, measured by not only the higher probability of him suffering a car accident but also by the higher probability that the other individual $j \neq i$ suffers a car accident because of the speed s^i of individual i.

(c) *Comparison.* Show that drivers have individual incentives to drive too fast, relative to the socially optimal speed, i.e., show that $\widehat{s}^i > \overline{s}^i$.

- Comparing expressions (1) and (2), yields

$$\frac{\partial v^1(\hat{s}^1)}{\partial s^1} < \frac{\partial v^1(\bar{s}^1)}{\partial s^1}$$

Since $\frac{\partial^2 v^i(s^i)}{(\partial s^i)^2} < 0$, we can confirm that the speed that individual 1 independently selects, $\widehat{s}^1$, is excessive from a social point of view, i.e., $\widehat{s}^1 > \overline{s}^1$. Similarly, comparing (1) and (3), we have that $\widehat{s}^2 > \overline{s}^2$. Intuitively, every driver does not internalize the negative externality that his speed imposes on other drivers (in the form of a higher probability of suffering a car accident) when he independently selects his driving speed.

- Figure 9.1 represents the marginal utility, $\frac{\partial v^i(s^i)}{\partial s^i}$, and marginal expected costs, individual marginal costs, $\frac{\partial \gamma}{\partial s^i}c^i$, and social marginal costs, $\frac{\partial \gamma}{\partial s^i}(c^i+c^j)$, to support the above explanation. Since the social marginal cost curve is higher for any speed level s^i than the individual marginal cost curve, the former crosses the marginal utility curve at a lower speed level, i.e., $\overline{s}^i < \widehat{s}^i$. Intuitively, the social planner internalizes the externality that additional speed imposes on other drivers (who could suffer a car accident due to the speed of driver i), and thus reduces the speed of both drivers.

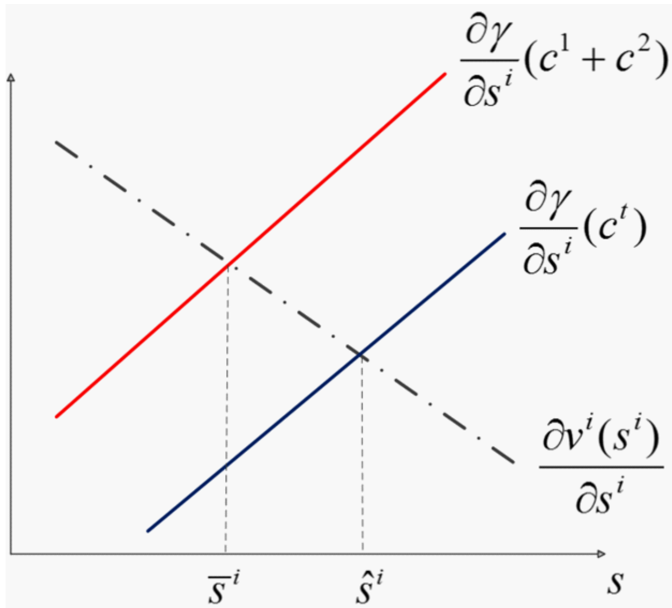

Figure 9.1. Efficient and socially optimal speed.

  - Note that for simplicity, we consider that the marginal utility decreases in s^i at a constant rate, i.e., $\frac{\partial^2 v^i(s^i)}{\partial {s^i}^2}$ is constant in s^i or, alternatively, $\frac{\partial^3 v^i(s^i)}{(\partial s^i)^3} = 0$; implying that the marginal utility curve is a straight line. In addition, we also assume that further increases in speed s^i imply a constant increase in the probability of an accident, i.e., $\frac{\partial^2 \gamma}{(\partial s^i)^2} > 0$ but constant or, alternatively, that $\frac{\partial^3 \gamma}{(\partial s^i)^3} = 0$. This property entails the marginal cost curve is also a straight line.
  - *Parametric example.* Continuing with the previous example in which $v^i(s^i) = \sqrt{s^i}$ and $\gamma(s^i, s^j) = \beta_i s^i + \beta_j s^j$, the socially optimal speed that the social planner would select, $\overline{s}^i$, is that satisfying

$$\frac{1}{2\sqrt{\overline{s}^i}} = \beta_i(c^i + c^j)$$

and solving for $\overline{s}^i$ yields $\overline{s}^i = \frac{1}{4[\beta_i(c^i+c^j)]^2}$, which clearly falls below the speed level independently selected by every driver $\widehat{s}^i = \frac{1}{4(\beta_i c^i)^2}$.

(d) *Restoring the social optimum.* Let us now evaluate the effect of speeding tickets (fines) to individuals driving too fast, i.e., to those drivers with a speed $\widehat{s}^i$

satisfying, $\widehat{s}^i > \overline{s}^i$. What is the dollar amount of the fine m^i that induces every individual i to fully internalize the externality he imposes onto others?

- Comparing (1) and (2) for driver 1, we must impose a fine of $m^1 = c^2$ in order to guarantee that (1) coincides with (2). Intuitively, this fine induces driver 1 to internalize the negative externality (higher chances of suffering a car accident and, in this case, an associated monetary cost of repairs) that he imposes on driver 2. Similarly comparing (1) and (3) for driver 2, we must impose a fine of $m^2 = c^1$ in order to guarantee that (1) coincides with (3).

(e) Let us now consider that individuals obtain a utility from driving fast, $v^i(s^i)$, only in the case that no accident occurs. Repeat steps (a)-(c), finding the optimal fine m^i that induces individuals to fully internalize the externality.

- *Equilibrium speed.* In this section of the exercise, driver i only obtains utility from driving fast, $v^i(s^i)$, when no accident occurs. Given that the probability that an accident does *not* occur is $1 - \gamma(s^1, s^2)$, the utility of driver i is

$$\underbrace{\left[1 - \gamma(s^1, s^2)\right] \left(v^i(s^i) + \alpha w^i\right)}_{\text{No accident}} + \underbrace{\gamma(s^i, s^j)(\alpha w^i - c^i)}_{\text{Accident}}$$

which can be rearranged as

$$v^i(s^i) + \alpha w^i - \gamma(s^i, s^j)\left[c^i + v^i(s^i)\right]$$

Taking first order conditions with respect to s^i, we obtain that the individual driver i independently selects the speed $\overline{s}^i$ that solves

$$\frac{\partial v^i(s^i)}{\partial s^i}\left[1 - \gamma(s^i, s^j)\right] = \frac{\partial \gamma}{\partial s^i}(c^i + v^i(s^i)) \tag{4}$$

where conveniently separates the marginal utility of driving faster in the left-hand side, which only arises if driver i does not suffer a car accident, an event with probability $1 - \gamma(s^i, s^j)$; and its associated marginal cost in the right-hand side, which captures the higher probability of suffering a car accident, $\frac{\partial \gamma}{\partial s^i}$, and its two costs: one explicit, c^i, and one implicit, namely, the utility from driving that driver i would have to give up (since he can only benefit from driving when he does not suffer a car accident).

  – *Parametric example.* Following with the on-going parametric example, the above first order condition (4) becomes

$$\frac{1}{2\sqrt{\overline{s}^i}}\left[1 - (\beta_i \overline{s}^i + \beta_j \overline{s}^j)\right] = \beta_i(c^i + \sqrt{\overline{s}^i}),$$

  and similarly for driver j. Before solving for $\overline{s}^i$ in order to driver i's best response function, let us assume (in order to keep our parametric example compact) that $\beta_i = \beta_j = \frac{1}{2}$ and $c^i = c^j = \frac{2}{3}$. In this context, solving for $\overline{s}^i$ we obtain

$$\overline{s}^i(\overline{s}^j) = 0.56 + \frac{2}{3\sqrt{112 - 27\overline{s}^j}}$$

  Since both drivers are symmetric, $\overline{s}^i = \overline{s}^j$, we can solve for $\overline{s}^i$ yielding a symmetric equilibrium speed level of $\overline{s}^i = 0.71$.

- *Socially optimal speed.* The social planner's maximization problem in this case becomes

$$\max_{s^1,s^2}\; v^1(s^1)+\alpha w^1-\gamma(s^1,s^2)\left[c^1+v^1(s^1)\right]+v^2(s^2)+\alpha w^2-\gamma(s^1,s^2)\left[c^2+v^2(s^2)\right]$$

Taking first order conditions with respect to s^i, we obtain that the socially optimal speed, $\widehat{s}^i$, solves

$$\frac{\partial v^i(s^i)}{\partial s^i}\left[1-\gamma(s^i,s^j)\right]=\frac{\partial\gamma}{\partial s^i}\left[c^i+v^i(s^i)\right]+\frac{\partial\gamma}{\partial s^i}\left[c^j+v^j(s^j)\right] \tag{5}$$

- *Comparison.* Comparing expressions (4) and (5), we obtain that the fine m^i that induces every individual i to internalize the externality that his driving imposes on others is

$$m^i=c^j+v^j(s^j)$$

Intuitively, now an increase in the speed of driver i not only increases the probability that driver j suffers a car accident, and thus needs to incur a cost of c^j, it also reduces the utility from driving that driver j can only experience if he is not involved in a car accident.

  – *Parametric example.* Following with the on-going parametric example, the above first order condition (4) becomes

$$\frac{1}{2\sqrt{\widehat{s}^{ii}}}\left[1-(\beta_i\widehat{s}^i+\beta_j\widehat{s}^j)\right]=\beta_i\left[(c^i+\sqrt{\widehat{s}^i})+(c^j+\sqrt{\widehat{s}^j})\right],$$

  Before solving for $\widehat{s}^i$ in order to driver i's best response function, let us assume (in order to keep our parametric example compact) that $\beta_i=\beta_j=\frac{1}{2}$ and $c^i=c^j=\frac{2}{3}$. In this context, we can simultaneously solve for $\widehat{s}^i$ and $\widehat{s}^j$ obtaining $\widehat{s}^i=\widehat{s}^j=0.26$, which is indeed a lower speed than when drivers independently choose their own driving speed, $\overline{s}^i=0.71$.

2. [**Flexible and Inflexible Environmental Policy.**] Consider an industry with an incumbent monopolist in period $t=1$ and a duopoly (i.e., the incumbent and an entrant) in period $t=2$. For simplicity, assume that both firms face the same constant marginal cost $c>0$, and a linear inverse demand curve $p(Q)=1-Q$, where Q represent aggregate output. Their output generates an environmental externality given by the convex damage function $ED(Q)=d\cdot Q^2$, where $d>0$. Assume that the social welfare function that the environmental protection agency (EPA) considers is

$$SW=CS+PS+T-ED$$

where CS (PS) denotes consumer (producer) surplus, respectively, and $T\equiv t\cdot Q$ represents total revenue from emission fees.[1]

[1] This exercise is based on Espinola-Arredondo et al. (2014), which extend the model to a setting of incomplete information between firms in order to evaluate whether environmental regulation can entail more entry-deterring effects when such policy is flexible or inflexible.

(a) *Flexible policy.* Assume that the EPA can easily adjust emission fees between the first and second period. Find the emission fee it sets to the monopolist in period 1, t_1, and to the duopolists in period 2, t_2.

- Since firms' optimal production decisions do not influence each other in the two periods and EPA can set emission fees independently for both periods, we can analyze the two periods separately.
- *First period.* In period 1, the monopolist maximizes its profit

$$\max_{Q_m} \quad (1 - Q_m) Q_m - (c + t_1) Q_m$$

which yields an optimal output function $Q_m = \frac{1-(c+t_1)}{2}$. EPA seeks to induce an output level that maximizes social welfare,

$$\max_{Q} \quad CS(Q) + PS(Q) + T(Q) - ED(Q)$$

where $CS(Q) = \frac{1}{2}Q^2$, $PS(Q) = (1-Q)Q - (c+t_1)Q$, $T(Q) = t_1 \cdot Q$, $ED(Q) = d \cdot Q^2$. Taking first order conditions with respect to Q, we obtain the socially optimal output $Q^{SO} = \frac{1-c}{1+2d}$. Then the emission fee t_1 can be solved by setting $Q_m = Q^{SO}$, that is

$$\frac{1-(c+t_1)}{2} = \frac{1-c}{1+2d}$$

which yields

$$t_1 = (2d-1)\frac{1-c}{1+2d}, \text{ or } t_1 = (2d-1) Q_m^{SO}$$

- *Second period.* In period 2, the incumbent solves

$$\max_{q_{inc}} \quad (1 - q_{inc} - q_{ent}) q_{inc} - (c + t_2) q_{inc}$$

whereas the entrant solves

$$\max_{q_{ent}} \quad (1 - q_{inc} - q_{ent}) q_{ent} - (c + t_2) q_{ent} - F$$

where F denotes its entry cost. Simultaneously solving the two problems we obtain the duopolists' optimal production

$$q_{inc} = q_{ent} = \frac{1-c-t_2}{3}$$

EPA seeks to induce a social optimal output level Q that solves

$$\max_{Q} \quad CS(Q) + PS(Q) + T(Q) - ED(Q)$$

which coincides with the EPA's problem when dealing with a single firm, and thus yields the same socially optimal output $Q^{SO} = \frac{1-c}{1+2d}$. However, such aggregate output must now produced by two firms, that is, we need that $q_{inc} + q_{ent} = Q^{SO}$ or more explicitly

$$\frac{1-c-t_2}{3} + \frac{1-c-t_2}{3} = \frac{1-c}{1+2d}$$

solving for the emission fee t_2, we find that the emission fee that induces the incumbent and entrant to produce a socially optimal aggregate output is

$$t_2 = \frac{4d-1}{2} \cdot \frac{1-c}{1+2d}, \text{ or } t_2 = \frac{4d-1}{2} \cdot \ Q^{SO}$$

(b) *Inflexible policy.* Assume now that the EPA cannot adjust environmental regulation after industry conditions change. Such inflexibility may be due to the institutional setting requiring that changes in environmental regulation must be approved by Congress. Find the unique emission fee t that the EPA sets across both time periods. [*Hint*: The EPA anticipates that such a policy will generate inefficiencies in one (or both) periods, but seeks to minimize the sum of such inefficiencies. For simplicity, assume no time discounting.]

- Any emission fee t deviating from the optimal emission fees in either period will cause inefficiencies. Let's find the optimal unique emission fee t by minimizes the sum of the deadweight losses across both periods.

$$\min_{t\geq 0} \quad DWL_1(t) + DWL_2(t)$$

where

$$DWL_1(t) = \int_{Q_m(t)}^{Q^{SO}} SMW(Q)\,dQ \quad \text{and} \quad DWL_2(t) = \int_{Q_{duo}(t)}^{Q^{SO}} SMW(Q)\,dQ,$$

and in the intervals of integration $Q^{SO} = \frac{1-c}{1+2d}$ denotes socially optimal output (which applies in both periods), $Q_m(t) = \frac{1-(c+t)}{2}$ represents the incumbent's output function under monopoly, and $Q_{duo}(t) = Q_{inc}(t) + Q_{ent}(t) = 2 \times \frac{1-c-t}{3}$ is the aggregate output function under duopoly. In addition, SMW represents social marginal welfare (the first order derivative of the EPA's welfare function with respect to Q), that is

$$\begin{aligned} SMW(Q) &= Q + 1 - 2Q - (c+t) + t - 2dQ \\ &= 1 - c - (1+2d)\,Q \end{aligned}$$

- Therefore, the deadweight loss in the first period is

$$\begin{aligned} DWL_1(t) &= \int_{Q_m(t)}^{Q^{SO}} SMW(Q)\,dQ \\ &= (1-c)\,Q^{SO} - \frac{1}{2}(1+2d)\left(Q^{SO}\right)^2 \\ &\quad - \left[(1-c)\,Q_m(t) - \frac{1}{2}(1+2d)\left(Q_m(t)\right)^2\right] \\ &= \frac{\left[(2d-1)c + 1 + t - 2d(1-t)\right]^2}{8(1+2d)} \end{aligned}$$

where we use $Q^{SO} = \frac{1-c}{1+2d}$, $Q_m(t) = \frac{1-(c+t)}{2}$, and $Q_{duo}(t) = Q_{inc}(t) + Q_{ent}(t) = 2 \times \frac{1-c-t}{3}$ in the last equality. Similarly, the deadweight loss in the second period is

$$\begin{aligned} DWL_2(t) &= \int_{Q_{duo}(t)}^{Q^{SO}} SMW(Q)\,dQ \\ &= (1-c)\,Q^{SO} - \frac{1}{2}(1+2d)\left(Q^{SO}\right)^2 \\ &\quad - \left[(1-c)\,Q_{duo}(t) - \frac{1}{2}(1+2d)\left(Q_{duo}(t)\right)^2\right] \\ &= \frac{\left[(4d-1)c+2+2t-4d(1-t)-1\right]^2}{18(1+2d)} \end{aligned}$$

- The regulator can now construct the sum $DWL_1(t) + DWL_2(t)$ (note that both $DWL_1(t)$ and $DWL_2(t)$ are strictly positive), as follows

  $$\min_{t\geq 0} \frac{\left[(2d-1)c+1+t-2d(1-t)\right]^2}{8(1+2d)} + \frac{\left[(4d-1)c+2+2t-4d(1-t)-1\right]^2}{18(1+2d)}$$

  Taking first-order conditions with respect to t, obtaining an inflexible emission fee of

  $$t = \frac{(1-c)\left[50d-17\right]}{25(1+2d)}.$$

  In order to confirm that the emission fee t indeed yields the minimum of the objective function $DWL_1(t) + DWL_2(t)$, note that such a function is convex in t, that is,

  $$\frac{\partial^2\left[DWL_1(t) + DWL_2(t)\right]}{\partial t^2} = \frac{25(1+2d)}{36} > 0$$

  for all parameter values.

(c) *Comparison.* Compare the flexible emission fees you found in part (a) with the inflexible fee found in part (b). Interpret.

- The inflexible emission fee t we just found can be expressed as a convex combination of the equilibrium fees under a flexible policy, t_1 and t_2, by solving $t = \alpha t_1 + (1-\alpha)t_2$, where parameter α describes the relative weight on first-period taxes. Solving for parameter α in

  $$t = \alpha t_1 + (1-\alpha)t_2$$

  that is,

  $$\underbrace{\frac{(1-c)\left[50d-17\right]}{25(1+2d)}}_{t} = \underbrace{\alpha(2d-1)\frac{1-c}{1+2d}}_{t_1} + (1-\alpha)\underbrace{\frac{4d-1}{2}\frac{1-c}{1+2d}}_{t_2}$$

  yields $\alpha = \frac{9}{25}$. Hence, $t = \frac{9}{25}t_1 + \frac{16}{25}t_2$, and thus $t_1 < t < t_2$, i.e., the inflexible fee t lies in between the flexible fee in the first and second period. From our analysis of the flexible policy, we know that fee t_2 is positive and induces positive output levels from both firms in the industry. Therefore, a lower fee t in the inflexible policy regime must also induce positive production levels from both incumbent and entrant.

3. [**Entry in the commons**] Consider a common pool resource initially operated by a single firm during two periods, appropriating x_i units in the first period and q_i units in the second period. In particular, assume that its first-period cost function is $\frac{x_i^2}{\theta}$ where $\theta > 0$, while second-period cost function is $\frac{q_i^2}{\theta-(1-\beta)x_i}$. Intuitively, parameter θ reflects the initial abundance of stock, i.e., a large θ decreases the firms' first and second-period costs; while β denotes the regeneration rate of the resource. Hence, if regeneration is complete, $\beta = 1$, first- and second-period costs coincide, but if regeneration is null, $\beta = 0$, second period costs become $\frac{q_i^2}{\theta-x_i}$ and thus every unit of first-period appropriation x_i increases the firm's second-period costs. For simplicity, assume that every unit of output is sold at a price of \$1 at the international market.[2]

   (a) Assuming no entry during both periods (i.e., the incumbent operates alone in both periods), find the profit-maximizing second-period appropriation, q_i^{NE}, and its first-period appropriation, x_i^{NE}, where superscript NE denotes no entry. [*Hint*: Use backward induction.]

      - *Second period.* Operating by backward induction, let us first analyze the second period. For a given first period appropriation x_i, with price equal to 1, the incumbent's profit maximization problem is

$$\max_{q_i \geq 0} \quad q_i - \frac{q_i^2}{\theta - (1-\beta)\, x_i}$$

        Taking first order conditions with respect to q_i yields

$$1 - \frac{2q_i}{\theta - (1-\beta)\, x_i} = 0$$

        Solving for q_i we obtain

$$q_i\left(x_i\right) = \frac{\theta - (1-\beta)\, x_i}{2}$$

        which is increasing in the initial abundance of the stock, θ, and in its regeneration rate, β, but decreasing in first-period appropriation, x_i.

      - *First period.* Given the optimal second-period appropriation function $q_i\left(x_i\right)$ we found above, the incumbent selects the level of x_i to solve the discounted sum of first and second periods profits.

$$\max_{x_i \geq 0} \quad \underbrace{\left[x_i - \frac{x_i^2}{\theta}\right]}_{\text{First period}} + \delta \underbrace{\left[q_i\left(x\right) - \frac{q_i^2\left(x\right)}{\theta - (1-\beta)\, x_i}\right]}_{\text{Second period}}$$

[2] This exercise is based on Espinola-Arredondo and Munoz-Garcia (2013). The exercise, however, focuses on a complete information setting, whereas the article examines how the presence of incomplete information affect equilibrium appropriation, and ultimately welfare levels.

Taking first order conditions with respect to x_i yields

$$\frac{1}{4}\left(4-\delta(1-\beta)-\frac{8x_i}{\theta}\right)=0$$

and solving for x_i we obtain

$$x_i^{NE}=\frac{\theta\left(4-(1-\beta)\,\delta\right)}{8}$$

Therefore, evaluating $q_i^{NE}(x_i)$ at x_i^{NE} yields a second-period appropriation of

$$\begin{aligned}
q_i^{NE}\left(x_i^{NE}\right) &= \frac{\theta-(1-\beta)\,x_i}{2}\\
&= \frac{\theta-(1-\beta)\cdot\frac{\theta(4-(1-\beta)\delta)}{8}}{2}\\
&= \frac{\theta\left[4+4\beta+(1-\beta)^2\,\delta\right]}{16}
\end{aligned}$$

(b) Assume that entry occurs in the second period, and that the second-period cost function for both incumbent and entrant becomes $\frac{(q_i+q_j)q_i}{\theta-(1-\beta)x_i}$. Find the profit-maximizing second-period appropriation, q_i^E and q_j^E, and first-period appropriation, x_i^E, where superscript E denotes entry.

- *Second period.* When entry occurs in the second period, the incumbent's second-period profit-maximization problem becomes

  $$\max_{q_i\geq 0}\quad q_i-\frac{(q_i+q_j)\,q_i}{\theta-(1-\beta)\,x_i}$$

  Taking first-order conditions and solving for q_i, we obtain the incumbent's best response function

  $$q_i\left(q_j,x_i\right)=\frac{\theta-(1-\beta)\,x_i}{2}-\frac{1}{2}q_j$$

  Note that when $q_j=0$, this function reduces to $q_i(0,x_i)=\frac{\theta-(1-\beta)x_i}{2}$, thus coinciding with the second-period appropriation level when entry does not ensue that we found in part (a). However, if $q_j>0$, the incumbent's second-period appropriation decreases.
- By symmetry, the entrant's best response function is

  $$q_j\left(q_i,x_i\right)=\frac{\theta-(1-\beta)\,x_i}{2}-\frac{1}{2}q_i$$

  Simultaneously solving for q_i and q_j, yields

  $$q_i^E\left(x_i\right)=q_j^E\left(x_i\right)=\frac{\theta-(1-\beta)\,x_i}{3}$$

- *First period.* Given these profit-maximizing second-period appropriation functions, the incumbent selects x_i in order to maximize the discounted sum of profits

$$\max_{x_i \geq 0} \quad \underbrace{\left[x_i - \frac{x_i^2}{\theta}\right]}_{\text{First period}} + \delta \underbrace{\left[q_i^E(x) - \frac{\left[q_i^E(x) + q_j^E(x)\right] q_i^E(x)}{\theta - (1-\beta) x_i}\right]}_{\text{Second period}}$$

  Taking first-order conditions with respect to x_i, yields

$$1 - \delta \frac{1}{9}(1-\beta) - \frac{2x_i}{\theta} = 0$$

  Solving for x_i we obtain the first-period appropriation in equilibrium

$$x_i^E = \frac{\theta\left[9 - (1-\beta)\delta\right]}{18}$$

  We can finally substitute x_i^E into $q_i^E(x_i)$ and $q_j^E(x_i)$, which yields a second-period appropriation level of

$$\begin{aligned} q_i^E\left(x_i^E\right) &= q_j^E\left(x_i^E\right) = \frac{\theta - (1-\beta)\left(\frac{\theta[9-(1-\beta)\delta]}{18}\right)}{3} \\ &= \frac{\theta\left[9 + \delta + \beta\left(9 - (2-\beta)\delta\right)\right]}{54} \end{aligned}$$