EconS 501 - Homework #6

Answer Key

Exercise from NS, 12.10.

a. Market equilibrium requires quantity demanded (evaluated at the price consumers pay) to equal quantity supplied (evaluated at the price producers receive):

\[ Q_D(P_D) = Q_S(P_S). \]

Substituting the relationship between consumer and producer prices \( P_D = (1+t)P_S \), we have

\[ Q_D((1+t)P_S) = Q_S(P_S). \]

Rearranging,

\[ 0 = Q_S(P_S) - Q_D((1+t)P_S). \]

Totally differentiating this identity with respect to \( t \),

\[ 0 = Q_S' \frac{dP_S}{dt} - Q_D' \left[ P_S + (1+t) \frac{dP_S}{dt} \right]. \]

The formulae we are asked to verify are only exact for an infinitesimal tax increase above an initial tax of \( t = 0 \). So throughout the remainder of the analysis, we will use the approximation \( t \approx 0 \). At these small tax levels, it is approximately true that \( P_S \approx P_D \) and \( Q_S \approx Q_D \).

Given that \( t \approx 0 \), the previous total derivative becomes

\[ 0 = Q_S' \frac{dP_S}{dt} - Q_D' \left( P_S + \frac{dP_S}{dt} \right). \]

Solving for \( \frac{dP_S}{dt} \),

\[ \frac{dP_S}{dt} = \frac{Q_D'P_S}{Q_S' - Q_D'}. \]

This implies

\[ \frac{d\ln P_S}{dt} = \frac{\frac{dP_S}{dt}}{P_S} = \frac{Q_D'}{Q_S' - Q_D'} = \frac{Q_D'(P_D/Q_D)}{Q_S'(P_S/Q_S) - Q_D'(P_D/Q_D)} = \frac{e_D}{e_S - e_D}. \]

The second to last line uses the approximations \( P_S \approx P_D \) and \( Q_S \approx Q_D \).

We can use a shortcut to find \( d\ln P_D/dt \). Totally differentiating the identity \( P_D = (1+t)P_S \), we obtain
\[
\frac{dP_D}{dt} = P_S + (1 + t) \frac{dP_S}{dt}
\]
\[
\approx P_S + \frac{dP_S}{dt},
\]
where the last line uses our approximations \( t \approx 0 \) and \( P_S \approx P_D \) for tiny tax rates. Hence
\[
\frac{d \ln P_D}{dt} = \frac{dP_D}{dt} \cdot \frac{1}{P_D}
\]
\[
\approx 1 + \frac{dP_S}{dt} \cdot \frac{1}{P_S}
\]
\[
= 1 + \frac{d \ln P_S}{dt}
\]
\[
= 1 + \frac{e_D}{e_s - e_D}
\]
\[
= \frac{e_s}{e_s - e_D}.
\]

b. \( DW \) is given as the area of the shaded region in the graph below. For a small tax increase starting from \( t = 0 \), \( DW \) can be approximated using the formula for the area of a triangle. (This is only an approximation because the supply and demand curves may not be straight lines.). Thus
\[
DW = \frac{1}{2} \left[ P_S (1 + t) - P_S \right] (Q_S - Q_o)
\]
\[
= \frac{t P_S}{2} (Q_S - Q_o)
\]
\[
\approx \frac{t P_S}{2} \left[ Q'_S \frac{dP_S}{dt} (-t) \right],
\]
or, rearranging,
\[
DW = -\frac{t^2}{2} (Q'_S P_s) \frac{dP_S}{dt}.
\]
As shown in part (a),
\[
\frac{dP_S}{dt} \cdot \frac{1}{P_S} = \frac{d \ln P_S}{dt} = \frac{e_D}{e_s - e_D}
\]
\[
\Rightarrow \frac{dP_S}{dt} = P_S \cdot \frac{e_D}{e_s - e_D} \approx P_0 \cdot \frac{e_D}{e_s - e_D}.
\]
The last approximation is good for a small tax increase above 0, implying \( P_S \approx P_0 \). Further, manipulating the expression to have an elasticity show up,
\[ Q_S P_s = Q_S P_s \left( \frac{Q_s}{Q_s} \right) \]
\[ = e_s Q_s \]
\[ \approx e_s Q_0, \]
where the approximation \( Q_s \approx Q_0 \) is again good for a small tax increase above 0. Substituting these results into the expression for \( DW \),
\[ DW \approx \frac{1^2}{2} \frac{e_d e_s}{e_s - e_d} P_s Q_0, \]
as was to be shown.

c. The unit tax described in this chapter is equivalent to the value of the ad-valorem tax. In other words, the unit tax is equal to the ad-valorem tax multiplied by \( P_s \). Therefore, the results obtained using the ad-valorem tax are equivalent to the ones obtained using the unit tax.
Exercises from MWG

Exercise 10.C.6

(a) If the specific tax $t$ is levied on the consumer, then he pays $p + t$ for every unit of the good, and the demand at market price $p$ becomes $x(p + t)$. The equilibrium market price $p^c$ is determined from equalizing demand and supply:

$$x(p^c + t) = q(p^c).$$

On the other hand, if the specific tax $t$ is levied on the producer, then he collects $p - t$ from every unit of the good sold, and the supply at market price $p$ becomes $q(p - t)$. The equilibrium market price $p^p$ is determined from equalizing demand and supply:

$$x(p^p) = q(p^p - t).$$

It is easy to see that $p$ solves the first equation if and only if $p + t$ solves the second one. Therefore, $p^p = p^c + t$, which is the ultimate cost of the good to consumers in both cases. The amount purchased in both cases is $x(p^p) = x(p^c + t)$.

(b) If the ad valorem tax $\tau$ is levied on the consumer, then he pays $(1 + \tau)p$ for every unit of the good, and the demand at market price $p$ becomes $x((1 + \tau)p)$. The equilibrium market price $p^c$ is determined from equalizing demand and supply:

$$x((1 + \tau)p^c) = q(p^c). \tag{1}$$

On the other hand, if the ad valorem tax $\tau$ is levied on the producer, collects $(1 - \tau)p$ from then he pays $(1 + \tau)p$ for every unit of the good sold, and the supply at market price $p$ becomes $q((1 - \tau)p)$. The equilibrium market price $p^p$ is determined from equalizing demand and supply:

$$x(p^p) = q((1 - \tau)p^p). \tag{2}$$

Consider the excess demand function for this case:

$$z(p) = x(p) - q((1 - \tau)p).$$
Since $x(\cdot)$ is non-increasing and $q(\cdot)$ is non-decreasing, $z(p)$ must be non-increasing. From (1) we have

$$
z((1 + \tau)p^c) = x((1 + \tau)p^c) - q(1 - \tau)((1 + \tau)p^c)] \\
= x((1 + \tau)p^c) - q((1 - \tau^2)p^c) \\
\geq x((1 + \tau)p^c) - q(p^c) = 0.
$$

taking into account that $q(\cdot)$ is non-decreasing and using (1).

Therefore, $z((1 + \tau)p^c) \geq 0$ and $z(p^P) = 0$. Since $z(\cdot)$ is non-decreasing, this implies that $(1 + \tau)p^c \leq p^P$. In words, levying the ad valorem tax on consumers leads to a lower cost on consumers than levying the same tax on producers. (In the same way it can be shown that levying the ad valorem tax on consumers leads to a higher price for producers than levying the same tax to producers).

If $q(\cdot)$ is strictly increasing, the argument can be strengthened to obtain the strict inequality: $(1 + \tau)p^c < p^P$. On the other hand, when the supply is perfectly inelastic, i.e. $q(p) = \bar{q} =$ const, then (1) and (2) combined yield $x((1 + \tau)p^c) = \bar{q} = x(p^P)$, and therefore $p^P = (1 + \tau)p^c$. Here both taxes result in the same cost to consumers. However, the producers still bear a higher burden when the tax is levied directly on them: $(1 - \tau)p^P = (1 - \tau)(1 + \tau)p^c < p^c$.

Therefore, the two taxes are still not fully equivalent.

The intuition behind these results is simple: with a tax, there is always a wedge between the “consumer price” and the “producer price”. Levying an ad valorem tax on the producer price, therefore, results in a higher tax burden (and a higher tax revenue) than levying the same percentage tax to the lower consumer price.

**Exercise 10.C.8.**

(a) Each firm’s profit can be written as

$$
\pi(q, \alpha) = p(\alpha)q - c(q, \alpha).
$$

The first-order condition of the firm’s profit-maximization problem can be written as

$$
p(\alpha) = c_q(q, \alpha). \quad \text{(FOC)}
$$
Denoting the solution of the firms’ problem by \( q^*(\alpha) \), we can write the firms’ reduced-form profits \( \pi^*(\alpha) = \pi(q^*(\alpha), \alpha) \).

Using the Envelop Theorem (in simple words, taking into account that \( \partial \pi(q, \alpha)/\partial q|_{q=q^*(\alpha)} = 0 \), we can write

\[
\pi^*(\alpha) = \frac{\partial \pi(q^*(\alpha), \alpha)}{\partial \alpha} = p'(\alpha)q^*(\alpha) - c_\alpha(q^*(\alpha), \alpha).
\]

(*)

By assumption \( c_\alpha(\cdot) \leq 0 \), and it is the first term, \( p'(\alpha)q^*(\alpha) \), which may present problems.

To determine how the market clearing price depends on \( \alpha \), consider the following system of two equations:

\[
p'(\alpha) = c_{qq}(q^*(\alpha), \alpha)q'_*(\alpha) + c_{qa}(q^*(\alpha), \alpha).
\]

\[
x'(p(\alpha))p'(\alpha) = Jq'_*(\alpha)
\]

The first equation is obtained by differentiating (FOC), the second by differentiating the market clearing condition \( x(p(\alpha)) = Jq^*(\alpha) \) (both with respect to \( \alpha \)). Eliminating \( q'_*(\alpha) \) from the system, we can solve for \( p'(\alpha) \):

\[
p'(\alpha) = c_{qa}/(1 - J^{-1}c_{qq}x'(p(\alpha)))
\]

(for convenience we have omitted arguments in the derivatives of \( c(\cdot) \)). Now we can rewrite (*) as

\[
\pi'_*(\alpha) = \frac{c_{qa}q^*(\alpha)}{1 - J^{-1}c_{qq}x'(p(\alpha))} - c_\alpha
\]

(b) The last expression can be rewritten as

\[
\pi'_*(\alpha) = \left[c_{qa}q^*(\alpha) - c_\alpha + c_\alpha J^{-1}c_{qq}x'(p(\alpha))\right] / [1 - J^{-1}c_{qq}x'(p(\alpha))].
\]

Remember that by assumption \( c_{qq} > 0 \) (costs are strictly convex in \( q \)), and \( x' \leq 0 \). Therefore, the denominator of the fraction is always positive, and

\[
\text{sign } \pi'_*(\alpha) = \text{sign}\left[c_{qa}q^*(\alpha) - c_\alpha + c_\alpha J^{-1}c_{qq}x'(p(\alpha))\right].
\]

(\text{**})

Since by assumption \( c_\alpha > 0 \), \( c_{qq} > 0 \), and \( x' \leq 0 \), the last term must be non-positive, and

\[
\text{sign } \pi'_*(\alpha) \leq \text{sign}\left[c_{qa}q^*(\alpha) - c_\alpha\right] = \text{sign } \partial \pi[\alpha] = \text{sign } \partial q[c_\alpha(q, \alpha)] \text{ at } q = q^*(\alpha).
\]
Thus, in order for profits to be increasing in \( \alpha \) for any demand function, is sufficient to have 
\[
\frac{\partial}{\partial q[c_{\alpha}(q,\alpha)]} \leq 0 \quad \text{at } q = q_*(\alpha).
\]

On the other hand, if this condition is not satisfied, we can take a demand function with \( |x'(p(\alpha))| \) sufficiently small. For such a demand function, the last term in the right-hand side of (***) will be very small, and we will have

\[
\text{sign} \eta'(\alpha) = \text{sign}[c_{qa}q_*(\alpha) - c_\alpha] > 0.
\]

i.e. profits increase in \( \alpha \).

(c) If \( \alpha \) is the price of factor input \( k \), then by Shepard’s lemma (Proposition 5.C.2(vi)) \( c_\alpha \) is the conditional demand for factor \( k \). Then the condition \( c_{qa} \leq 0 \) means that the conditional demand for \( k \) is non-increasing in output, i.e. that \( k \) is an inferior factor.

Exercise 10.C.10

The first-order condition for the firms’ profit-maximization problem can be written as \( p = c'(q) \). The market clearing condition can be written as \( x(p) = Jq \). Substituting the functional forms for \( c'(\cdot) \) and \( x(\cdot) \), we obtain the following system of equations:

\[
p = \beta q^\eta.
\]

\[
\alpha p^\varepsilon = Jq.
\]

Taking logs on both sides, we obtain a system of linear equations in \( \log p \) and \( \log q \). The solution of this system is

\[
\log p = (\log \beta + \eta \log \alpha - \eta \log J)/(1 - \varepsilon \eta),
\]

\[
\log q = (\varepsilon \log \beta + \log \alpha - \log J)/(1 - \eta).
\]

From here we can compute the elasticities:

\[
\frac{\partial \log p}{\partial \log \alpha} = \eta/(1 - \varepsilon \eta),
\]

\[
\frac{\partial \log p}{\partial \log \beta} = 1/(1 - \varepsilon \eta),
\]

\[
\frac{\partial \log q}{\partial \log \alpha} = 1/(1 - \varepsilon \eta),
\]

\[
\frac{\partial \log q}{\partial \log \beta} = \varepsilon/(1 - \eta).
\]
\[
\frac{\partial \log q}{\partial \log \beta} = \varepsilon / (1 - \eta).
\]

We see that increasing the elasticity of demand \(|\varepsilon|\) reduces the sensitivity of \(p\) and \(q\) to shifts in \(\alpha\), reduces the sensitivity of \(p\) to shifts in \(\beta\). Similarly, increasing the inverse elasticity of supply \(\eta\) reduces the sensitivity of \(p\) and \(q\) to shifts in \(\beta\), reduces the sensitivity of \(q\) to shifts in \(\alpha\), but increases the sensitivity of \(p\) to shifts in \(\alpha\).

**Exercise 10.F.3**

(Printing errata: You need to assume that taxes are small, which is necessary for a definite comparison. Also, the condition \(\phi''(\cdot) < 0\) in the third line of the exercise should instead be \(\phi''(\cdot) > 0\) ) Let \(p^*, q^*\), and \(J^*\) denote respectively the market price, each firm’s production, and the number of producing firms at the initial equilibrium. These variables should satisfy conditions (i)-(iii) on p.335 in the textbook.

Introduction of an ad valorem tax \(\tau\) can be represented by replacing the demand function \(x(p)\) with \(x(p(1 + \tau))\). This change leaves conditions (i) and (iii) intact. Therefore, the new long-run market price and firm output have to satisfy (i) and (iii), and thus have to coincide with \(p^*\) and \(q^*\). The new long-run equilibrium number of firms \(\hat{J}\) is determined from the modified (ii):

\[
x(p^*(1 + \tau)) = \hat{J}q^*
\]

Differentiating this expression with respect to \(\tau\) and substituting \(\tau = 0\), we obtain

\[
\hat{J}'(0) = p^*x'(p^*)/q^*
\]  \hspace{1cm} (1)

The resulting tax revenue is \(\hat{R}(\tau) = \tau p^* q^* \hat{J}\). The second-order Taylor expansion of this function around \(\tau = 0\) can be computed as

\[
\hat{R}(\tau) = \hat{R}'(0)\tau + \hat{R}''(0)\tau^2 = p^* q^*/(\tau J'(\tau) + J(\tau))|_{\tau=0}\tau + (\tau J''(\tau) + 2J'(\tau))|_{\tau=0}\tau^2/2
\]

\[
= p^* q^* J'(\tau) + p^* q^* J''(\tau)\tau^2
\]  \hspace{1cm} (2)

Introduction of a per firm tax \(T\), on the other hand, modifies the free entry condition (iii), so that now it can be written as \(p^0 q^0 - c(q^0) - T = 0\). The profit maximization condition (i) yields \(p^0 = c'(q^0)\). Combining with the previous equation and substituting our cost function we can write

\[
\phi'(q^0) = \left(\phi(q^0) + K + T\right)/q^0.
\]
This equation determines $q^0$. Differentiating it with respect to $T$, we obtain

$$q^0(T) = (\phi''(q^0)q^0)^{-1}.$$  

The number of firms $J^0$ can be determined from (ii) and the profit maximization condition:

$$x(\phi'(q^0)) = J^0 q^0.$$  

Differentiating this equation with respect to $T$ and evaluating at $T=0$, we obtain

$$J^0'(0) = [x'(\phi'(q^0))\phi''(q^0) - J^0]q^0(T)/q^0|_{T=0} = [x'(p^*)\phi''(q^*) - J^*/(\phi''(q^*)q^*)]. \tag{3}$$  

The tax revenue from the per firm tax is $R^0(T) = TJ^0(T)$. The second-order Taylor expansion of this function around $T = 0$ can be computed as

$$R^0(T) \approx R^0'(0)T + R^0''(0)T^2/2 = (TJ^0'(T) + J^0(T))|_{T=0}T + (TJ^0''(T) + 2J^0'(T))|_{T=0}T^2/2 = J^*T + J^0'(0)T^2$$  

Since at the initial equilibrium the two taxes raise the same revenue, we must have $T = \tau p^* q^*$. Substituting in the last Taylor expansion, we obtain

$$R^0(T) = p^* q^* J^* \tau + (p^* q^*)^2 J^0'(0)\tau^2 \tag{4}$$  

Comparing (2) and (4), we see that the first-order terms in $\tau$ coincide, and the second-order difference can be computed using (1) and (3):

$$R^0(T) - \tilde{R}(\tau) \approx p^* q^* / p^* q^* J^0(0) - \tilde{J}'(0)/\tau^2$$  

$$= p^* q^* [p^*x'(p^*)/q^* - p^* J^*/(\phi''(q^*)q^*)] - p^* x'(q^*)/q^* \tau^2 = -p^* p^2 J^*/\phi''(q^*) < 0.$$  

Therefore, a small ad valorem tax raises more revenue than the corresponding per firm tax.