EconS 501 - Homework #6

Answer Key

Exercise from NS, 12.10.

a. Market equilibrium requires quantity demanded (evaluated at the price consumers pay) to equal quantity supplied (evaluated at the price producers receive):

$$Q_D(P_D) = Q_S(P_S).$$

Substituting the relationship between consumer and producer prices $P_D = (1+t)P_S$, we have

$$Q_D((1+t)P_S) = Q_S(P_S).$$

Rearranging,

$$0 = Q_S(P_S) - Q_D((1+t)P_S).$$

Totally differentiating this identity with respect to t,

$$0 = Q_S' \frac{dP_S}{dt} - Q_D' \left[P_S + (1+t)\frac{dP_S}{dt} \right].$$

The formulae we are asked to verify are only exact for an infinitesimal tax increase above an initial tax of $t = 0$. So throughout the remainder of the analysis, we will use the approximation $t \approx 0$. At these small tax levels, it is approximately true that $P_S \approx P_D$ and $Q_S \approx Q_D$.

Given that $t \approx 0$, the previous total derivative becomes

$$0 = Q_S' \frac{dP_S}{dt} - Q_D' \left(P_S + \frac{dP_S}{dt} \right).$$

Solving for dP_S/dt,

$$\frac{dP_S}{dt} = \frac{Q_D' P_S}{Q_S' - Q_D'}.$$

This implies

$$\begin{aligned} \frac{d \ln P_S}{dt} &= \frac{dP_S}{dt} \cdot \frac{1}{P_S} \\ &= \frac{Q_D'}{Q_S' - Q_D'} \\ &= \frac{Q_D' \left(P_D/Q_D \right)}{Q_S' \left(P_S/Q_S \right) - Q_D' \left(P_D/Q_D \right)} \\ &= \frac{e_D}{e_S - e_D} \end{aligned}$$

The second to last line uses the approximations $P_S \approx P_D$ and $Q_S \approx Q_D$.

We can use a shortcut to find $d \ln P_D/dt$. Totally differentiating the identity $P_D = (1+t)P_S$, we obtain

$$\frac{dP_D}{dt} = P_S + (1+t)\frac{dP_S}{dt}$$
$$\approx P_D + \frac{dP_S}{dt},$$

where the last line uses our approximations $t \approx 0$ and $P_S \approx P_D$ for tiny tax rates. Hence

$$\frac{d \ln P_D}{dt} = \frac{dP_D}{dt} \cdot \frac{1}{P_D}$$
$$\approx 1 + \frac{dP_S}{dt} \cdot \frac{1}{P_S}$$
$$= 1 + \frac{d \ln P_S}{dt}$$
$$= 1 + \frac{e_D}{e_S - e_D}$$
$$= \frac{e_S}{e_S - e_D}.$$

b. DW is given as the area of the shaded region in the graph below. For a small tax increase starting from $t = 0$, DW can be approximated using the formula for the area of a triangle. (This is only an approximation because the supply and demand curves may not be straight lines.). Thus

$$DW = \frac{1}{2}\left[P_S(1+t) - P_S\right]\left(Q_S - Q_0\right)$$
$$= \frac{tP_S}{2}\left(Q_S - Q_0\right)$$
$$\approx \frac{tP_S}{2}\left[Q_S' \frac{dP_S}{dt}\left(-t\right)\right],$$

or, rearranging,

$$DW = -\frac{t^2}{2}\left(Q_S' P_S\right)\frac{dP_S}{dt}.$$

As shown in part (a),

$$\frac{dP_S}{dt} \cdot \frac{1}{P_S} = \frac{d \ln P_S}{dt} = \frac{e_D}{e_S - e_D}$$
$$\Rightarrow \frac{dP_S}{dt} = P_S \cdot \frac{e_D}{e_S - e_D} \approx P_0 \cdot \frac{e_D}{e_S - e_D}.$$

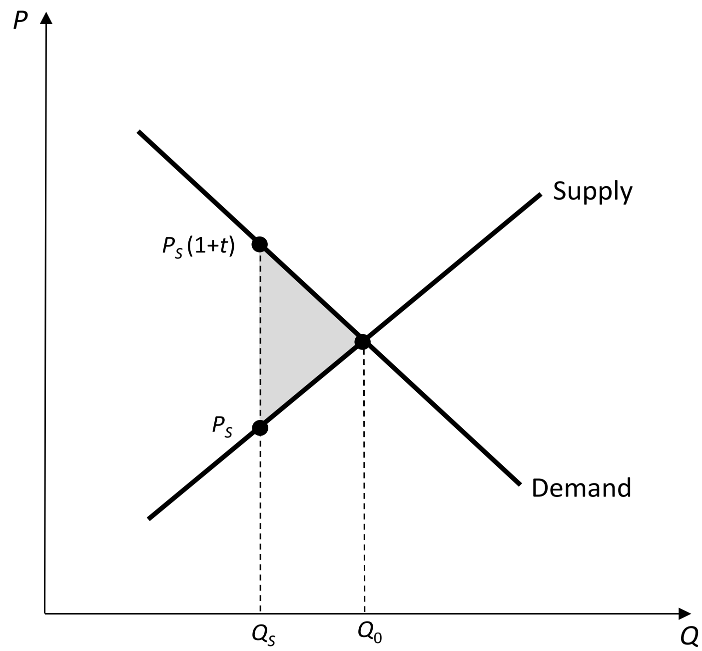
The last approximation is good for a small tax increase above 0, implying $P_S \approx P_0$. Further, manipulating the expression to have an elasticity show up,

$$\begin{aligned} Q_S' P_S &= Q_S' P_S \left(\frac{Q_S}{Q_S} \right) \\ &= e_S Q_S \\ &\approx e_S Q_0, \end{aligned}$$

where the approximation $Q_S \approx Q_0$ is again good for a small tax increase above 0. Substituting these results into the expression for DW,

$$DW \approx \frac{t^2}{2} \frac{e_D e_S}{e_S - e_D} P_0 Q_0,$$

as was to be shown.



c. The unit tax described in this chapter is equivalent to the value of the ad-valorem tax. In other words, the unit tax is equal to the ad-valorem tax multiplied by P_s. Therefore, the results obtained using the ad-valorem tax are equivalent to the ones obtained using the unit tax.

Exercises from MWG

Exercise 10.C.6

(a) If the specific tax t is levied on the consumer, then he pays $p + t$ for every unit of the good, and the demand at market price p becomes $x(p+t)$. The equilibrium market price p^c is determined from equalizing demand and supply:

$$x(p^c + t) = q(p^c).$$

On the other hand, if the specific tax t is levied on the producer, then he collects $p - t$ from every unit of the good sold, and the supply at market price p becomes $q(p-t)$. The equilibrium market price p^p is determined from equalizing demand and supply:

$$x(p^p) = q(p^p - t).$$

It is easy to see that p solves the first equation if and only if $p + t$ solves the second one. Therefore, $p^p = p^c + t$, which is the ultimate cost of the good to consumers in both cases. The amount purchased in both cases is $x(p^p) = x(p^c + t)$.

(b) If the ad valorem tax τ is levied on the consumer, then he pays $(1+\tau)p$ for every unit of the good, and the demand at market price p becomes $x((1+\tau)p)$. The equilibrium market price p^c is determined from equalizing demand and supply:

$$x\big((1+\tau)p^c\big) = q(p^c). \tag{1}$$

On the other hand, if the ad valorem tax τ is levied on the producer, collects $(1-\tau)p$ from then he pays $(1+\tau)p$ for every unit of the good sold, and the supply at market price p becomes $q((1-\tau)p)$. The equilibrium market price p^p is determined from equalizing demand and supply:

$$x(p^p) = q\big((1-\tau)p^p\big). \tag{2}$$

Consider the excess demand function for this case:

$$z(p) = x(p) - q\big((1-\tau)p\big).$$

Since $x(\cdot)$ is non-increasing and $q(\cdot)$ is non-decreasing, $z(p)$ must be non-increasing. From (1) we have

$$z\big((1+\tau)p^c\big) = x\big((1+\tau)p^c\big) - q\big((1-\tau)[(1+\tau)p^c]\big)$$

$$= x\big((1+\tau)p^c\big) - q\big((1-\tau^2)p^c\big)$$

$$\geq x\big((1+\tau)p^c\big) - q(p^c) = 0.$$

taking into account that $q(\cdot)$ is non-decreasing and using (1).

Therefore, $z((1+\tau)p^c) \geq 0$ and $z(p^p) = 0$. Since $z(\cdot)$ is non-increasing, this implies that $(1+\tau)p^c \leq p^p$. In words, levying the ad valorem tax on consumers leads to a lower cost on consumers than levying the same tax on producers. (In the same way it can be shown that levying the ad valorem tax on consumers leads to a higher price for producers than levying the same tax to producers).

If $q(\cdot)$ is strictly increasing, the argument can be strengthened to obtain the strict inequality: $(1+\tau)p^c < p^p$. On the other hand, when the supply is perfectly inelastic, i.e. $q(p) = \overline{q} = const$, then (1) and (2) combined yield $x\big((1+\tau)p^c\big) = \overline{q} = x(p^p)$, and therefore $p^p = (1+\tau)p^c$. Here both taxes result in the same cost to consumers. However, the producers still bear a higher burden when the tax is levied directly on them: $(1-\tau)p^p = (1-\tau)(1+\tau)p^c < p^c$.

Therefore, the two taxes are still not fully equivalent.

The intuition behind these results is simple: with a tax, there is always a wedge between the "consumer price" and the "producer price". Levying an ad valorem tax on the producer price, therefore, results in a higher tax burden (and a higher tax revenue) than levying the same percentage tax to the lower consumer price.

Exercise 10.C.8.

(a) Each firm's profit can be written as

$$\pi(q,\alpha) = p(\alpha)q - c(q,\alpha).$$

The first-order condition of the firm's profit-maximization problem can be written as

$$p(\alpha) = c_q(q,\alpha). \qquad \text{(FOC)}$$

Denoting the solution of the firms' problem by $q_*(\alpha)$, we can write the firms' reduced-form profits $\pi_*(\alpha) = \pi(q_*(\alpha), \alpha)$.

Using the Envelop Theorem (in simple words, taking into account that $\partial\pi(q,\alpha)/\partial q|_{q=q_*(\alpha)} = 0$), we can write

$$\pi_*'(\alpha) = \frac{\partial\pi(q_*(\alpha),\alpha)}{\partial\alpha} = p'(\alpha)q_*(\alpha) - c_\alpha(q_*(\alpha),\alpha). \qquad (*)$$

By assumption $c_\alpha(\cdot) \leq 0$, and it is the first term, $p'(\alpha)q_*(\alpha)$, which may present problems.

To determine how the market clearing price depends on α, consider the following system of two equations:

$$p'(\alpha) = c_{qq}(q_*(\alpha),\alpha)q_*'(\alpha) + c_{q\alpha}(q_*(\alpha),\alpha).$$

$$x'\big(p(\alpha)\big)p'(\alpha) = Jq_*'(\alpha)$$

The first equation is obtained by differentiating (FOC), the second by differentiating the market clearing condition $x\big(p(\alpha)\big) = Jq_*(\alpha)$ (both with respect to α). Eliminating $q_*'(\alpha)$ from the system, we can solve for $p'(\alpha)$:

$$p'(\alpha) = c_{q\alpha}/(1 - J^{-1}c_{qq}x'(p(\alpha)))$$

(for convenience we have omitted arguments in the derivatives of $c(\cdot)$). Now we can rewrite (*) as

$$\pi_*'(\alpha) = \frac{c_{q\alpha}q_*(\alpha)}{1 - J^{-1}c_{qq}x'\big(p(\alpha)\big)} - c_\alpha$$

(b) The last expression can be rewritten as

$$\pi_*'(\alpha) = \big[c_{q\alpha}q_*(\alpha) - c_\alpha + c_\alpha J^{-1}c_{qq}x'\big(p(\alpha)\big)\big]/[1 - J^{-1}c_{qq}x'(p(\alpha))].$$

Remember that by assumption $c_{qq} > 0$ (costs are strictly convex in q), and $x' \leq 0$. Therefore, the denominator of the fraction is always positive, and

$$sign\pi_*'(\alpha) = sign\big[c_{q\alpha}q_*(\alpha) - c_\alpha + c_\alpha J^{-1}c_{qq}x'\big(p(\alpha)\big)\big]. \qquad (**)$$

Since by assumption $c_\alpha > 0, c_{qq} > 0$, and $x' \leq 0$, the last term must be non-positive, and

$$sign\pi_*'(\alpha) \leq sign\big[c_{q\alpha}q_*(\alpha) - c_\alpha\big] = sign\partial/\partial q[c_\alpha(q,\alpha)] \;\; at\ q = q_*(\alpha).$$

Thus, in order for profits to be increasing in α for any demand function, is sufficient to have $\frac{\partial}{\partial q[c_\alpha(q,\alpha)]} \leq 0 \;\; at \; q = q_*(\alpha)$.

On the other hand, if this condition is not satisfied, we can take a demand function with $|x'(p(\alpha))|$ sufficiently small. For such a demand function, the last term in the right-hand side of (**) will be very small, and we will have

$$sign\pi_*'(\alpha) = sign\big[c_{q\alpha}q_*(\alpha) - c_\alpha\big] > 0.$$

i.e. profits increase in α.

(c) If α is the price of factor input *k*, then by Shepard's lemma (Proposition 5.C.2(vi)) c_α is the conditional demand for factor *k*. Then the condition $c_{q\alpha} \leq 0$ means that the conditional demand for *k* is non-increasing in output, i.e. that *k* is an inferior factor.

Exercise 10.C.10

The first-order condition for the firms' profit-maximization problem can be written as $p = c'(q)$. The market clearing condition can be written as $x(p) = Jq$. Substituting the functional forms for $c'(\cdot)$ and $x(\cdot)$, we obtain the following system of equations:

$$p = \beta q^\eta.$$

$$\alpha p^\varepsilon = Jq.$$

Taking logs on both sides, we obtain a system of linear equations in log *p* and log *q*. The solution of this system is

$$\log p = (\log \beta + \eta \log \alpha - \eta \log J)/(1 - \varepsilon\eta),$$

$$\log q = (\varepsilon \log \beta + \log \alpha - \log J)/(1 - \varepsilon\eta).$$

From here we can compute the elasticities:

$$\partial \log p \,/\partial \log \alpha = \eta/(1 - \varepsilon\eta),$$

$$\partial \log p \,/\partial \log \beta = 1/(1 - \varepsilon\eta),$$

$$\partial \log q \,/\partial \log \alpha = 1/(1 - \varepsilon\eta),$$

$$\partial \log q / \partial \log \beta = \varepsilon/(1-\varepsilon\eta).$$

We see that increasing the elasticity of demand $|\varepsilon|$ reduces the sensitivity of p and q to shifts in α, reduces the sensitivity of p to shifts in β. Similarly, increasing the inverse elasticity of supply η reduces the sensitivity of p and q to shifts in β, reduces the sensitivity of q to shifts in α, but increases the sensitivity of p to shifts in α.

Exercise 10.F.3

(Printing errata: You need to assume that taxes are small, which is necessary for a definite comparison. Also, the condition $\phi''(\cdot) < 0$ in the third line of the exercise should instead be $\phi''(\cdot) > 0$.) Let p^*, q^*, and J^* denote respectively the market price, each firm's production, and the number of producing firms at the initial equilibrium. These variables should satisfy conditions (i)-(iii) on p.335 in the textbook.

Introduction of an ad valorem tax τ can be represented by replacing the demand function $x(p)$ with $x(p(1+\tau))$. This change leaves conditions (i) and (iii) intact. Therefore, the new long-run market price and firm output have to satisfy (i) and (iii), and thus have to coincide with p^* and q^*. The new long-run equilibrium number of firms $\hat{J}$ is determined from the modified (ii):

$$x\big(p^*(1+\tau)\big) = \hat{J}q^*$$

Differentiating this expression with respect to τ and substituting $\tau = 0$, we obtain

$$\hat{J}'(0) = p^* x'(p^*)/q^* \tag{1}$$

The resulting tax revenue is $\hat{R}(\tau) = \tau p^* q^* \hat{J}$. The second-order Taylor expansion of this function around $\tau = 0$ can be computed as

$$\begin{aligned}\hat{R}(\tau) = \hat{R}'(0)\tau + \hat{R}''(0)\tau^2 &= p^*q^*/(\tau\hat{J}'(\tau) + \hat{J}(\tau))|_{\tau=0}\tau + (\tau\hat{J}''(\tau) + 2\hat{J}'(\tau))|_{\tau=0}\tau^2/2 \\ &= p^*q^*J^*\tau + p^*q^*\hat{J}'(\tau)\tau^2 \end{aligned} \tag{2}$$

Introduction of a per firm tax T, on the other hand, modifies the free entry condition (iii), so that now it can be written as $p^0q^0 - c(q^0) - T = 0$. The profit maximization condition (i) yields $p^0 = c'(q^0)$. Combining with the previous equation and substituting our cost function we can write

$$\phi'(q^0) = (\phi(q^0) + K + T)/q^0.$$

This equation determines q^0. Differentiating it with respect to *T*, we obtain

$$q^{0\prime}(T) = (\phi''(q^0)q^0)^{-1}.$$

The number of firms J^0 can be determined from (ii) and the profit maximization condition:

$$x\big(\phi'(q^0)\big) = J^0 q^0.$$

Differentiating this equation with respect to *T* and evaluating at *T*=0, we obtain

$$J^{0\prime}(0) = [x'\big(\phi'(q^0)\big)\phi''(q^0) - J^0]q^{0\prime}(T)/q^0|_{T=0} = [x'(p^*)\phi''(q^*) - J^*]/(\phi''(q^*)q^{*2}). \quad (3)$$

The tax revenue from the per firm tax is $R^0(T) = TJ^0(T)$. The second-order Taylor expansion of this function around $T = 0$ can be computed as

$$R^0(T) \approx R^{0\prime}(0)T + R^{0\prime\prime}(0)T^2/2 = (TJ^{0\prime}(T) + J^0(T))|_{T=0}T + (TJ^{0\prime\prime}(T) + 2J^{0\prime}(T))|_{T=0}T^2/2$$
$$= J^*T + J^{0\prime}(0)T^2$$

Since at the initial equilibrium the two taxes raise the same revenue, we must have $T = \tau p^* q^*$. Substituting in the last Taylor expansion, we obtain

$$R^0(T) = p^*q^*J^*\tau + (p^*q^*)^2 J^{0\prime}(0)\tau^2 \quad (4)$$

Comparing (2) and (4), we see that the first-order terms in τ coincide, and the second-order difference can be computed using (1) and (3):

$$R^0(T) - \hat{R}(\tau) \approx p^*q^*/p^*q^*J^{0\prime}(0) - \hat{J}'(0)/\tau^2$$
$$= p^*q^*[p^*x'(p^*)/q^* - p^*J^*/(\phi''(q^*)q^*) - p^*x'(q^*)/q^*]\tau^2 = -p^{*2}\tau^2 J^*/\phi''(q^*) < 0.$$

Therefore, a small ad valorem tax raises more revenue than the corresponding per firm tax.