Micro Theory I
Assignment #5 - Answer key

1. **Exercises from MWG (Chapter 6):**

   (a) **Exercise 6.B.1 from MWG:** Show that if the preferences $\succsim$ over $\mathcal{L}$ satisfy the independence axiom, then for all $\alpha \in (0,1)$ and $L, L', L'' \in \mathcal{L}$ we have

   $$L \succ L' \text{ if and only if } \alpha L + (1-\alpha) L'' \succ \alpha L' + (1-\alpha) L''$$

   and

   $$L \sim L' \text{ if and only if } \alpha L + (1-\alpha) L'' \sim \alpha L' + (1-\alpha) L''.$$

   Show also that if $L \succ L'$ and $L'' \succ L'''$, then $\alpha L + (1-\alpha) L'' \succ \alpha L' + (1-\alpha) L'''$.

- – Suppose first that $L \succ L'$. A first application of the independence axiom (in the "only-if" direction in Definition 6.B.4) yields

  $$\alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''.$$

  If these two compound lotteries were indifferent, then a second application of the independence axiom (in the "if" direction) would yield $L' \succsim L$, which contradicts $L \succ L'$. We must thus have

  $$\alpha L + (1-\alpha) L'' \succ \alpha L' + (1-\alpha) L''.$$

  Suppose conversely that

  $$\alpha L + (1-\alpha) L'' \succ \alpha L' + (1-\alpha) L'',$$

  then, by the independence axiom $L \succsim L'$. If these two simple lotteries were indifferent, then the independence axiom would imply

  $$\alpha L' + (1-\alpha) L'' \succsim \alpha L + (1-\alpha) L'',$$

  a contradiction. We must thus have $L \succ L'$.

  – Suppose next that $L \sim L'$, then $L \succsim L'$ and $L' \succsim L$. Hence by applying the independence axiom twice (in the "only if" direction), we obtain

  $$\alpha L + (1-\alpha) L'' \sim \alpha L' + (1-\alpha) L''.$$

  Conversely, we can show that if

  $$\alpha L + (1-\alpha) L'' \sim \alpha L' + (1-\alpha) L'',$$

  then $L \sim L'$.

- For the last part of the exercise, suppose that $L \succ L'$ and $L'' \succ L'''$, then, by independence axiom and the first assertion of this exercise,

$$\alpha L + (1-\alpha) L'' \succ \alpha L' + (1-\alpha) L''$$

and

$$\alpha L' + (1-\alpha) L'' \succ \alpha L' + (1-\alpha) L'''.$$

Thus, by the transitivity of $\succ$ (Proposition 1.B.1(i)),

$$\alpha L + (1-\alpha) L'' \succ \alpha L' + (1-\alpha) L'''.$$

b. **Exercise 6.B.4 from MWG:**

- We can assign utility levels (u_A, u_B, u_C, u_D) so that $u_A = 1$ and $u_D = 0$ as a normalization (see Proposition 6.B.2). Then $u_B = p \times 1 + (1-p) \times 0 = p$ and $u_C = q \times 1 + (1-q) \times 0 = q$.
- The probability distribution under Criterion 1 is

$$(p_A, p_B, p_C, p_D) = (0.891, 0.099, 0.009, 0.001)$$

and the probability distribution under Criterion 2 is

$$(p_A, p_B, p_C, p_D) = (0.8415, 0.1485, 0.0095, 0.0005)$$

The expected utility under Criterion 1 is thus

$$0.891 + 0.099p + 0.009q$$

The expected utility under Criterion 2 is thus

$$0.8415 + 0.1485p + 0.0095q$$

Hence, the agency would prefer Criterion 1 if and only if p and q satisfy $99 > 99p + q$ and it would prefer Criterion 2 if and only if $99 > 99p + q$

2. **Exercise 4 from Lecture 7 (Rubinstein):**

(a) A decision maker is to choose an action from a set A. The set of consequences is Z. For every action $a \in A$, the consequence z^* is realized with probability α and any $z \in Z \setminus z^*$ is realized with probability $r(a, z) = (1-\alpha)q(a, z)$. Assume that after making his choice he is told that z^* will not occur and is given a chance to change his decision. Show that if the decision maker obeys the Bayesian updating rule and follows vNM axioms, he will not change his decision.

- By the vNM Theorem, preferences exhibit expected utility representation. Before learning the information, the decision maker solves

$$\max_{a \in A} \sum_{z \in Z \setminus z^*} r(a, z)\, v(z) + \alpha v(z^*).$$

After learning that z^* will not occur, the decision maker updates his beliefs so that

$$r^{'}(a,z) = \frac{r(a,z)}{1-\alpha} = q(a,z)$$

for all $z \in Z \setminus z^*$ and the decision maker solves

$$\max_{a \in A} \sum_{z \in Z \setminus z^*} r^{'}(a,z)\, v(z),$$

which yields the same solution. Intuitively, the decision maker cannot affect the probability of outcome z^* occuring, since it happen with probability p for all effort levels e. Hence, his optimal choice of a (e.g., effort) is unaffected by the information he received. (Of course, this result would not apply if outcome z^* was affected by the level of a chosen by the decision maker, i.e., if $r(a,z)$ was non-constant in a.

(b) Give an example where a decision maker who follows nonexpected utility preference relation or obeys a non-Bayesian updating rule is not time consistent.

- **Example 1.** Assume the decision maker has a "worst case" preference relation, where z_1 is the best prize, z_2 is the second best, and z^* is the worst. Let action a_1 yield z_1 for sure, and action a_2 yield z_1 and z_2 with equal probability, conditional on z^* not occurring. Then the decision maker will initially be indifferent between a_1 and a_2, but will strictly prefer a_1 after the information is revealed.
- **Example 2.** Assume that $Z = \{1, 2, 3\}$, that $z^* = 0$, and that the Bernouilli utility function is linear, $v(z) = z$. Assume that initially his beliefs are: $q(a_1, 2) = 1$, $q(a_2, 3) = 0.4$ and $q(a_2, 1) = 0.6$. In words, when the decision maker chooses action a_1, he believes that outcome 2 occurs with certainty; whereas when he chooses action a_2, he believes that outcome 3 occurs with probability 0.4 and outcome 1 happens with the remaining probability (0.6). Contingentally the decision maker chooses a_1. If he updates his beliefs and after he was lucky to avoid z^* he believes that he will be fortunate again, that is $q^{'}(a_2, 3) = 1$, then he will change his mind and choose a_2.

3. **Exercises from Rubinstein:**

   (a) Lecture 7 (Expected utility): Exercises 4 and 5.

   (b) Lecture 8 (Risk): Exercise 6.

   - See answer key at the end of this handout.

4. **[Investment]** An individual is an expected utility maximizer described by the intertemporally additive preference-scaling function

$$u(c_0) + \beta u(c_1)$$

where $u(\cdot)$ is a strictly concave function with $u''(\cdot)$. The individual has current income I_0. The individual can buy bonds at unit price $p = \beta$ which pay out in the next period one unit of consumption per unit of bond held.

(a) Compare the individual's demand for bonds in the case where her future income is certain and equal to I_0, and the situation in which there is a 50% chance that her future income is $I_0 - \varepsilon$ and a 50% chance that her future income is $I_0 + \varepsilon$.

(b) Show that the individual's demand for bonds (i.e., for "saving") is greater when she faces uncertain future income than when she faces certain future income.

- See answer key at the end of this handout.

5. **[Hyperbolic Absolute Risk Aversion, HARA]** Consider the family of utility functions with Hyperbolic Absolute Risk Aversion (HARA) as follows

$$u(x) = \frac{1}{\beta - 1}(\alpha + \beta x)^{\frac{\beta-1}{\beta}},$$

where $\beta \neq 0$ and $\beta \neq 1$. Find the Arrow-Pratt coefficient of absolute risk-aversion, $r_A(x, u)$. Describe how it varies in parameters α and β.

- First, note that the first derivative of this utility function is $u'(x) = (\alpha + \beta x)^{-\frac{1}{\beta}}$, while the second derivative is $u''(x) = -(\alpha + \beta x)^{-\frac{1+\beta}{\beta}}$. Figure 1 depicts this function for different values of β.

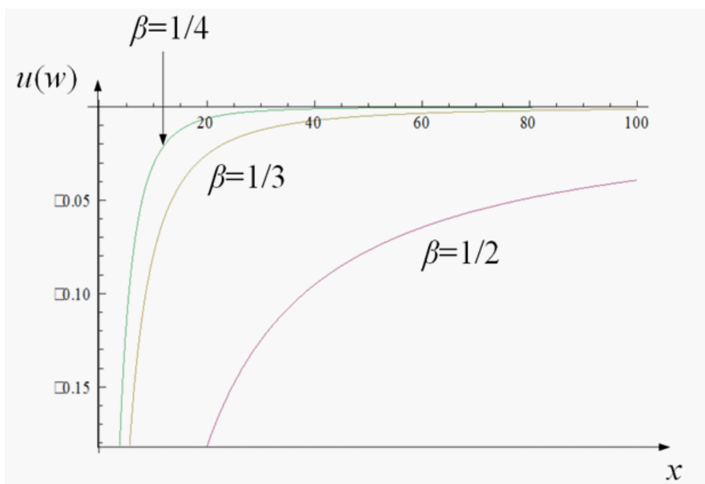

Figure 1. HARA utility function.

- The Arrow-Pratt coefficient of absolute risk-aversion is

$$r_A(x, u) = -\frac{u''(x)}{u'(x)} = -\frac{-(\alpha + \beta x)^{-\frac{1+\beta}{\beta}}}{(\alpha + \beta x)^{-\frac{1}{\beta}}} = \frac{1}{\alpha + \beta x},$$

which is decreasing in wealth, x, as long as $\beta > 0$, but it is increasing if $\beta < 0$. Figure 2 depicts the Arrow-Pratt coefficient of absolute risk aversion for different values of parameter β.[1]

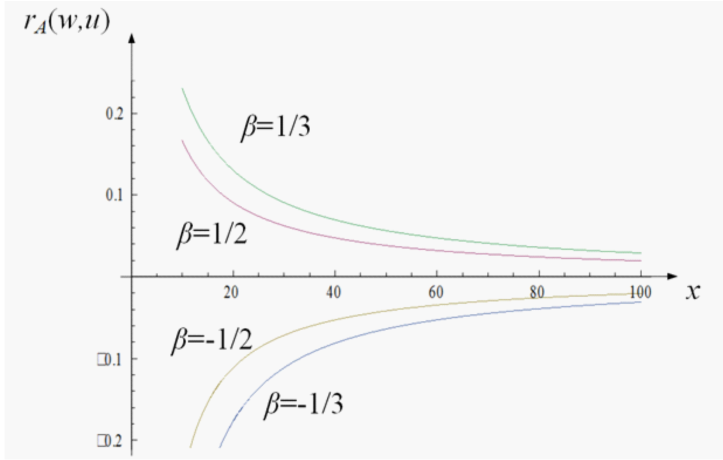

Figure 2. $r_A(x, u)$ for the HARA utility function.

6. [[**Non-constant coefficient of absolute risk aversion**] Suppose that the utility function is given by

$$u(w) = aw - bw^2,$$

where $a, b > 0$, and $w > 0$ denotes income.

(a) Find the coefficient of absolute risk-aversion, $r_A(w, u)$. Does it increase or decrease in wealth? Interpret.

- First, note that $u' = a - 2bw$ and $u'' = -2b$. Hence, the Arrow-Pratt coefficient of absolute risk-aversion is

$$r_A(w, u) = -\frac{u''(w)}{u'(w)} = \frac{2b}{a - 2bw}$$

Note that, as w rises, the denominator decreases, and as a consequence $r_A(w, u)$ rises, i.e., the decision maker becomes more risk averse as his wealth increases.

- Importantly, this exercise illustrates that, while the decision maker can have a concave utility function (indeed, $u'' = -2b < 0$, as illustrated in Figure 3, which depicts utility function $u(w) = aw - bw^2$ evaluated at parameters $a = 80$ and $b = 1$), the Arrow-Pratt coefficient of absolute risk aversion,

[1] For more information on the HARA utility function, including behavioral patterns in different investment settings, see its wikipedia entry at the following link: http://en.wikipedia.org/wiki/Hyperbolic_absolute_risk_aversion, and the references included in the link.

$r_A(w,u)$, can increase as he becomes richer.

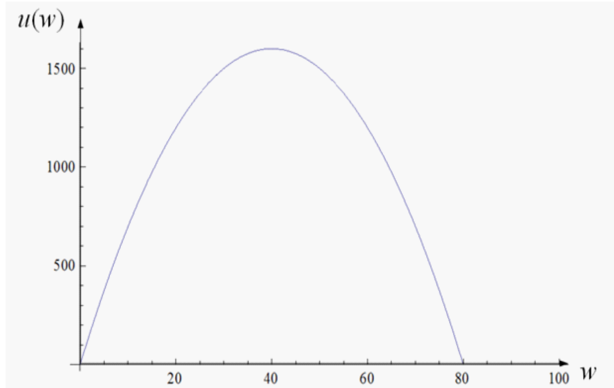

Figure 3. Utility function $u(w) = aw - bw^2$

(b) Let us now consider that this decision maker is deciding how much to invest in a risky asset. This risky asset is a random variable R, with mean $\overline{R} > 0$ and variance σ_R^2. Assuming that his initial wealth is w, state the decision maker's expected utility maximization problem, and find first order conditions.

- First, note that the decision maker's wealth (W in his utility function) is now a random variable $w + xR$, where x is the amount of risky asset that he acquires. Inserting this expression in the decision maker's utility function, and taking expectations we obtain that the decision maker selects his optimal investment in risky asset, x, in order to solve

$$\max_{x} \ E\left[a\left(w + xR\right) - b\left(w + xR\right)^2\right]$$

Taking first order conditions with respect to x, we obtain

$$E\left[aR - 2bR\left(w + x^* R\right)\right] = 0$$

- We can use the definition of the variance of random variable R, $\sigma_R^2 = E[R^2] - \overline{R}^2$, to obtain $E[R^2] = \overline{R}^2 + \sigma_R^2$. Hence, the above first order condition can be simplified to

$$\begin{aligned} E\left[aR - 2bR\left(w + x^* R\right)\right] &= a\overline{R} - 2b\overline{R}w - E\left[2bR^2 x^*\right] = \\ &= a\overline{R} - 2b\overline{R}w - 2bx^*\left(\overline{R}^2 + \sigma_R^2\right) = 0 \end{aligned}$$

(c) What is the optimal investment in risky assets?

- Solving for x^* in the above expression, $a\overline{R} - 2b\overline{R}w - 2bx^*\left(\overline{R}^2 + \sigma_R^2\right) = 0$, we obtain

$$x^* = \frac{\left(a - 2bw\right)\overline{R}}{2b\left(\overline{R}^2 + \sigma_R^2\right)}$$

(d) Show that the optimal amount of investment in risky assets is a decreasing function in wealth. Interpret.

- Differentiating x^* with respect to wealth, yields

$$\frac{\partial x^*}{\partial w} = -\frac{\overline{R}}{\left(\overline{R}^2 + \sigma_R^2\right)}$$

which is negative, since $\overline{R}$, $\sigma_R^2 > 0$. Intuitively, the larger the decision maker's wealth, the lower is the amount of risky assets he wants to hold. This explanation is consistent with his coefficient of absolute risk aversion found at the beginning of the exercise, where we showed that the individual becomes more risk averse as his wealth increases.

EconS 501 – Homework #5 – Answer Key

Rubinstein – Lecture 8 – Problem 6

Assume there are a finite number of income levels. An income distribution specifies the proportion of individuals at each level. Thus, an income distribution has the same mathematical structure as a lottery. Consider the binary relation "one distribution is more egalitarian than another".

Alternative interpretation: *Another way to think of this is to imagine that babies are born into a rich, poor, or somewhere in-between family in the US.*

Part (a). Why is the von Neumann-Morgenstern independence axiom inappropriate for characterizing this type of relation?

ANSWER:

First (verbal) intuition: *Continuing our babies example. All babies in the society being born into rich families is as egalitarian as all of them being born into poor families. Under the independence axiom, half of babies being born into rich families and half being born into poor families should be as egalitarian as all of them being born into rich families, but our intuition is that half rich and half poor is less egalitarian than assigning all babies to families with the same income.*

A more formal answer: More generally, assigning all members of the society an income of \$1 is as egalitarian as assigning all of them an income of \$2, i.e., $1 \sim 2$. Using the independence axiom (multiplying each side of the indifference relation by $\frac{1}{2}$, and adding $\frac{1}{2}1$ on both sides of the indifference relation), we obtain

$$\frac{1}{2}1+\frac{1}{2}1 \sim \frac{1}{2}2+\frac{1}{2}1 .$$

This implies that a distribution where all individuals receive \$1 (see left-hand side of the above result) should be as egalitarian as lottery $\frac{1}{2}2+\frac{1}{2}1$. However, our intuition is that $\frac{1}{2}2+\frac{1}{2}1$ is *less* egalitarian than assigning the same income of \$1 to all members of the society.

Part (b). Suggest and formulate a property that is appropriate, in your opinion, as an axiom for this relation. Give two examples of preference relations that satisfy this property.

ANSWER: If *p* and *q* are identical distributions, except that the highest (lowest) income level in *p* is less (more) than in *q*, then *p* is more egalitarian than *q*. Less egalitarian because there's a greater chance that one person could get a very high assignment and another person could get a very low assignment.

Here are two examples of preference relations that satisfy this property:

1) $p \succsim q$ if $Var(p) \le Var(q)$, that is, distribution *p* is weakly preferred to *q* if its variance is lower.

2) $p \succsim q$ if $\max_{z \in \text{supp } p(z)} z - \min_{z \in \text{supp } p(z)} z \le \max_{z \in \text{supp } q(z)} z - \min_{z \in \text{supp } q(z)} z$, that is, distribution *p* is weakly preferred to *q* if the difference between the payoff of the richest and poorest individual is smaller in distribution *p* than in *q*.

Exercise #3 (Investment) – Homework #5

Let *q* denote the quantity of bonds the individual buys in period 0. Let c_1 be her consumption in period 1. Her maximization problem may be expressed as follows

$$\begin{aligned} &\max_q u(c_0) + \beta u(c_1) \\ &s.t.\ \ c_0 = I_0 - pq \\ &\qquad\ c_1 = I_0 q \end{aligned}$$

or plugging in for the unconstrained problem:

$$\max_q u(I_0 - pq) + \beta u(I_0 + q)$$

Taking first order conditions:

$$\frac{\partial}{\partial q} : -u'(I_0 - pq^*)p + \beta u'(I_0 + q^*) = 0$$

Since $p = \beta$, and *u* is strictly concave, it follows that the unique solution is $q^* = 0$.

Now let c_1 be her consumption in period 1 when her income is $I_0 + \varepsilon$ and let c_2 be her consumption in period 1 when her income is $I_0 - \varepsilon$. Her maximization problem may now be expressed as follows

$$\begin{aligned} &\max_q u(c_0) + \beta(\tfrac{1}{2}u(c_1) + \tfrac{1}{2}u(c_2)) \\ &s.t.\ \ \ c_0 = I_0 - pq \\ &\qquad\ \ c_1 = I_0 + \varepsilon + q \\ &\qquad\ \ c_2 = I_0 - \varepsilon + q \end{aligned}$$

Or plugging in for the unconstrained problem:

$$\max_q u(I_0 - pq) + \beta(\tfrac{1}{2}u(I_0 + \varepsilon + q) + \tfrac{1}{2}u(I_0 - \varepsilon + q))$$

Taking first order conditions:

$$\frac{\partial}{\partial q}: -u'(I_0 - pq^{**})p + \beta(\tfrac{1}{2}u'(I_0 + \varepsilon + q^{**}) + \tfrac{1}{2}u'(I_0 - \varepsilon + q^{**})) = 0$$

CLAIM: $q^{**} > q^* = 0$.

The claim is true: if we plug q = 0 into the left hand side of the first order condition we arrive at:

$$-u'(I_0)\beta + \beta(\tfrac{1}{2}u'(I_0 + \varepsilon) + \tfrac{1}{2}u'(I_0 - \varepsilon)) > \beta(-u'(I_0) + u'(\tfrac{1}{2}(I_0 + \varepsilon) + \tfrac{1}{2}(I_0 - \varepsilon)) = 0$$

The inequality follows from Jensen's inequality applied to a convex function (u' is a convex function since u''' >0). Thus we see that the individual's demand for bonds (can be thought of as demand for "saving") is greater when she faces uncertain future income than when she faces certain future income. This comparative static effect is an illustration of the "precautionary motive for saving". Mathematically it results because u' is convex.