EconS 501 - Microeconomic Theory I
Homework #4 - Answer key

1. **[Duality in production theory]** Consider a firm with production function $q = \sqrt{z}$, using one input (e.g., labor) to produce units of output q. The price of every unit of input is $w > 0$, and the price of every unit of output is $p > 0$.

   (a) Set up the firm's profit-maximization problem (PMP), and solve for its unconditional factor demand $z(w, p)$.

   - The firm chooses the units of input z to solve

   $$\max_{z \geq 0} p\sqrt{z} - wz$$

   where the first term indicates total revenue, whereas the second reflects total costs. Taking first-order condition with respect to z, we obtain

   $$p\frac{1}{2}z^{-1/2} - w \leq 0.$$

   In the case of interior solutions, we can solve for z to find the unconditional factor demand

   $$z(w, p) = \frac{p^2}{4w^2}.$$

   (b) What is the output level that arises from using the amount of inputs $z(w, p)$? Label this output level $q(w)$.

   - Inserting $z(w, p)$ into the firm's production function $\sqrt{z}$, we obtain

   $$q(w) = \frac{p}{2w}$$

   (c) Set up the firm's cost-minimization problem (CMP), and solve for its conditional factor demand $z(w, q)$ for any output level q. (For now, we write the constraint of the CMP to be $f(z) \geq q$, where the output level q that the firm seeks to reach does not necessarily coincide with that found in part (b), $q(w)$.)

   - The firm chooses the units of input z to solve

   $$\min_{z \geq 0} w \cdot z$$

   $$\text{subject to} \quad \sqrt{z} \geq q$$

   Setting up the Lagrangian, we obtain

   $$L = w \cdot z - \lambda\left(\sqrt{z} - q\right).$$

   Taking first-order condition with respect to z, we find that

   $$w - \frac{\lambda}{2\sqrt{z}} = 0,$$

and solving for z, we find

$$z = \frac{\lambda^2}{4w^2}.$$

Now, note that the constraint must be binding in equilibrium, so that $\sqrt{z} = q$. Otherwise, the firm could still reduce its total costs and satisfy the output constraint (reaching output target q). Using the binding constraint $\sqrt{z} = q$ into the above result, we obtain that

$$\lambda = 2qw$$

Last, we solve for z, to find the conditional factor demand

$$z(w, q) = q^2$$

(d) Evaluate the conditional factor demand $z(w, q)$ at output level $q = q(w)$, to obtain $z(w, q(w))$. Show that it coincides with the unconditional factor demand $z(w, p)$ found in part (a), that is,

$$z(w, q(w)) = z(w, p).$$

- We find that

$$z(w, q(w)) = \left(\frac{p}{2w}\right)^2 = \frac{p^2}{4w^2} = z(w, p)$$

which coincides with the unconditional factor demand $z(w, p)$ found in part (a).

(e) *Shephard's lemma.* Evaluate the CMP's objective function, $w \cdot z$, at the conditional factor demand $z(w, q)$, to obtain the cost function, that is, find $c(w, q) = w \cdot z(w, q)$. Differentiate the cost function with respect to w, and show that your result coincides with the conditional factor demand $z(w, q)$.

- The cost function is

$$c(w, q) = w \cdot z(w, q) = wq^2$$

Differentiating with respect to input price w, we obtain

$$\frac{\partial c(w, q)}{\partial w} = q^2$$

which coincides with the conditional factor demand $z(w, q)$ found in part (c).

(f) *Substitution and output effects.* Let us now consider that the firm faces cheaper wages (lower w). Differentiate the unconditional factor demand $z(w, p)$ found in part (a) with respect to w to find the total effect of this price change.

- Differentiating $z(w, p)$ with respect to input price w, we obtain

$$\frac{\partial z(w, p)}{\partial w} = -\frac{p^2}{2w^3}$$

which is negative, thus indicating that higher wages induce the firm to hire fewer workers.

(g) Differentiate the conditional factor demand $z(w, q)$ found in part (c) with respect to w to obtain the substitution effect of this price change.

- Differentiating $z(w, q)$ with respect to input price w, we obtain

$$\frac{\partial z(w,q)}{\partial w} = 0$$

In this case, this derivative reflects that, if the firm had to solve the CMP at the new input price (while still reaching the same output target q), it would have to choose same workers.

(h) Compare your results in parts (f) and (g). Which is the output effect of the change in w?

- Comparing $\frac{\partial z(w,p)}{\partial w}$ (which captures the total effect) and $\frac{\partial z(w,q)}{\partial w}$ (which only measures the substitution effect), we find that the output effect is

$$\frac{\partial z(w,p)}{\partial w} - \frac{\partial z(w,q)}{\partial w} = -\frac{p^2}{2w^3} - 0 = -\frac{p^2}{2w^3}$$

which is also negative. Hence, as wages increase, the firm chooses to produce fewer units, which ultimately reduces its factor demand (hiring fewer workers).

2. **[Convex cost function].** Consider a firm with Cobb-Douglas production function $q = f(z_1, z_2) = z_1^\alpha z_2^\beta$, where $\alpha, \beta > 0$. Assume that the firm faces input prices w_1 and w_2, both being positive.

(a) Set up the firm's cost minimization problem (CMP) and find the conditional factor demand correspondence, $z_i(w, q)$, for every input $i = \{1, 2\}$.

- The firm solves

$$\min_{z_1, z_2 \geq 0} \quad w_1 z_1 + w_2 z_2$$

$$\text{subject to} \quad f(z_1, z_2) = z_1^\alpha z_2^\beta \geq q$$

From similar exercises, we know that an interior solution exists where

$$MRTS_{1,2} = \frac{w_1}{w_2}$$

which yields

$$\frac{\alpha z_1^{\alpha-1} z_2^\beta}{\beta z_1^\alpha z_2^{\beta-1}} = \frac{w_1}{w_2}$$

or, after rearranging,

$$\frac{\alpha z_2}{\beta z_1} = \frac{w_1}{w_2}$$

which, solving for z_2, yields

$$z_2 = \frac{\beta z_1 w_1}{\alpha w_2}$$

Inserting this result into the binding output constraint, $z_1^\alpha z_2^\beta = q$, we find

$$z_1^\alpha \left(\frac{\beta z_1 w_1}{\alpha w_2} \right)^\beta = q$$

which, solving for z_1, yields the conditional factor demand correspondence for input 1,

$$z_1(w, q) = \left[q \left(\frac{\alpha w_2}{\beta w_1} \right)^\beta \right]^{-(\alpha+\beta)}$$

Inserting this result into the equation we found above, $z_2 = \frac{\beta z_1 w_1}{\alpha w_2}$, we obtain the conditional factor demand correspondence for input 2,

$$z_2(w, q) = q^{\frac{1}{\alpha+\beta}} \left(\frac{\beta w_1}{\alpha w_2} \right)^{\frac{\alpha}{\alpha+\beta}}$$

(b) Show that the conditional factor demand $z(w, q)$ satisfies homogeneity of degree zero in w.

- If we increase the price of all inputs, w_1 and w_2, by a common factor λ, the conditional factor demand correspondence for input 1 becomes

$$z_1(\lambda w, q) = \left[q \left(\frac{\alpha \lambda w_2}{\beta \lambda w_1} \right)^\beta \right]^{-(\alpha+\beta)} = \left[q \left(\frac{\alpha w_2}{\beta w_1} \right)^\beta \right]^{-(\alpha+\beta)} = z_1(w, q)$$

which simplifies to

$$z_1(\lambda w, q) = \left[q \left(\frac{\alpha w_2}{\beta w_1} \right)^\beta \right]^{-(\alpha+\beta)} = z_1(w, q)$$

thus coinciding with $z_1(w, q)$. In words, increasing the price of all inputs by a common factor λ does not change the firm's demand for inputs; although it makes such demand more expensive to purchase, as we show when analyzing the cost function in part (d) below. Similarly, if we increase the price of all inputs, w_1 and w_2, by a common factor λ, the conditional factor demand correspondence for input 2 becomes

$$z_2(\lambda w, q) = q^{\frac{1}{\alpha+\beta}} \left(\frac{\beta \lambda w_1}{\alpha \lambda w_2} \right)^{\frac{\alpha}{\alpha+\beta}}$$

which simplifies to

$$z_2(\lambda w, q) = q^{\frac{1}{\alpha+\beta}} \left(\frac{\beta w_1}{\alpha w_2} \right)^{\frac{\alpha}{\alpha+\beta}}$$

thus coinciding with $z_2(w, q)$.

(c) Find the cost function $c(w, q) = w_1 z_1(w, q) + w_2 z_2(w, q)$.

- The cost function is

$$
\begin{aligned}
c(w,q) &= w_1 z_1(w,q) + w_2 z_2(w,q) \\
&= w_1 \left[q \left(\frac{\alpha w_2}{\beta w_1} \right)^{\beta} \right]^{-(\alpha+\beta)} + w_2 q^{\frac{1}{\alpha+\beta}} \left(\frac{\beta w_1}{\alpha w_2} \right)^{\frac{\alpha}{\alpha+\beta}} \\
&= q^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha}{\beta} \right)^{\frac{-\alpha}{\alpha+\beta}} \frac{\alpha+\beta}{\beta} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}}
\end{aligned}
$$

(d) Show that the cost function $c(w,q)$ satisfies: (i) homogeneity of degree one in w; (ii) it is weakly increasing in output q; and (iii) $\frac{\partial c(w,q)}{\partial w_i} = z_i(w,q)$ for every input $i = \{1,2\}$ (recall that this last property is commonly referred as Shepard's lemma).

- *Homogeneity of degree one in input prices w.* If we increase the price of all inputs, w_1 and w_2, by a common factor λ, the cost function becomes

$$
c(\lambda w, q) = q^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha}{\beta} \right)^{\frac{-\alpha}{\alpha+\beta}} \frac{\alpha+\beta}{\beta} (\lambda w_1)^{\frac{\alpha}{\alpha+\beta}} (\lambda w_2)^{\frac{\beta}{\alpha+\beta}}
$$

which simplifies to

$$
q^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha}{\beta} \right)^{\frac{-\alpha}{\alpha+\beta}} \frac{\alpha+\beta}{\beta} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}} \lambda = \lambda c(w,q)
$$

thus confirming homogeneity of degree one in input prices w. In words, if we increase the price of all inputs by a common factor λ, the minimal cost that the firm needs to incur increases proportionally (exactly by factor λ).

- *Weakly increasing in output q.* Differentiating the cost function with respect to q, we find

$$
\frac{\partial c(w,q)}{\partial q} = \frac{1}{\beta} q^{\frac{1-\alpha-\beta}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha}{\beta} \right)^{\frac{-\alpha}{\alpha+\beta}} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}} > 0
$$

as required. Intuitively, if the firm seeks to produce a larger output level, it needs to incur a larger total cost.

- *Shepard's lemma.* Differentiating the cost function with respect to the price of input $i = \{1,2\}$, w_i, we obtain

$$
\frac{\partial c(w,q)}{\partial w_1} = \frac{\alpha+\beta}{\beta} \frac{\alpha}{\alpha+\beta} q^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha}{\beta} \right)^{\frac{-\alpha}{\alpha+\beta}} \left(\frac{w_2}{w_1} \right)^{\frac{\beta}{\alpha+\beta}} = z_1(w,q)
$$

$$
\frac{\partial c(w,q)}{\partial w_2} = \frac{\alpha+\beta}{\beta} \frac{\beta}{\alpha+\beta} q^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha}{\beta} \right)^{\frac{-\alpha}{\alpha+\beta}} \left(\frac{w_1}{w_2} \right)^{\frac{\alpha}{\alpha+\beta}} = z_2(w,q)
$$

thus coinciding with the conditional factor demand of input i.

(e) Show that if the production function $q = f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$ satisfies constant returns to scale (which occurs when $\alpha + \beta = 1$), then $z(w,q)$ and $c(w,q)$ are homogeneous of degree 1 in output q.

- If we increase the output level that the firm seeks to reach by λ, the conditional factor demand correspondence of input i becomes

$$z_1(w, \lambda q) = \lambda q \, (frac\alpha\beta)^{\beta} \left(\frac{w_2}{w_1}\right)^{\beta} = \lambda z_1(w, q)$$

$$z_2(w, \lambda q) = \lambda q \, (frac\alpha\beta)^{-\alpha} \left(\frac{w_2}{w_1}\right)^{-\alpha} = \lambda z_2(w, q)$$

which simplifies to $\lambda z_i(w, q)$ if $\alpha + \beta = 1$. In words, the firm increases its factor demand proportionally with the increase in output.

- Similarly, if we increase the output level that the firm seeks to reach by λ, the cost function becomes

$$c(w, \lambda q) = \lambda q \left(\frac{\alpha}{\beta}\right)^{-\alpha} \frac{1}{\beta} w_1^{\alpha} w_2^{\beta} = c(w, q)$$

which simplifies to $\lambda c(w, q)$ if $\alpha + \beta = 1$. In words, the firm increases its minimal cost of production proportionally with the increase in output.

(f) What happens with your results in part (e) if the production function exhibits increasing returns to scale ($\alpha + \beta > 1$)? What if it exhibits decreasing returns to scale ($\alpha + \beta < 1$)?

- Similarly, we can see that $z_i(w, \lambda q) < \lambda z_i(w, q)$ and $c(w, \lambda q) < \lambda c(w, q)$ if he production function exhibits increasing returns to scale ($\alpha + \beta > 1$). And vice versa if it exhibits decreasing returns to scale ($\alpha + \beta < 1$).

(g) Under which conditions is the production function $q = f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$ concave in input z? Under which conditions is the cost function $c(w, q)$ convex in output q?

- Taking derivative with respect to input z_1 we get

$$\frac{\partial q}{\partial z_1} = \alpha z_1^{\alpha-1} z_2^{\beta}$$

which is positive as long as $\alpha > 0$. For the second derivative, we have

$$\frac{\partial^2 q}{\partial z_1^2} = \alpha\left(\alpha - 1\right) z_1^{\alpha-2} z_2^{\beta}$$

which is negative (guaranteeing that the supply function is concave in input z_1) if and only if $\alpha < 1$. Similarly, for input z_2, we find

$$\frac{\partial q}{\partial z_2} = \beta z_1^{\alpha} z_2^{\beta-1} > 0 \quad \text{and} \quad \frac{\partial^2 q}{\partial z_2^2} = \beta\left(\beta - 1\right) z_1^{\alpha} z_2^{\beta-2} < 0$$

as long as $\beta > 0$ and $\beta < 1$, respectively. Hence, the supply function is concave in the input vector $z = (z_1, z_2)$ if and only if $\alpha < 1$ and $\beta < 1$.

- Following the same procedure for the cost function, we obtain

$$\frac{\partial^2 c(w,q)}{\partial q^2} = \frac{1-\alpha-\beta}{(\alpha+\beta)^2} q^{\frac{1-2\alpha-2\beta}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}}$$

which is positive (guaranteeing convex costs) if and only if $\alpha + \beta < 1$. [Note that this condition is less demanding than the pair of conditions found above, $\alpha < 1$ and $\beta < 1$. To see this, plot α on the horizontal and β on the vertical axis. Condition $\alpha < 1$ and $\beta < 1$ defines a square of side 1, whereas condition $\alpha + \beta < 1$ defines a diagonal originating at 1 and reaching 1 on the horizontal axis; thus encapsulating the square previously found.]

3. [**Scale of firms in a perfectly competitive market**] Consider an industry in which there is free entry, whereby all firms have the same U-shaped AC curve.

(a) Show that for each firm the marginal cost curve is

$$MC = \sum_{j=1}^{n} \frac{w_j z_j}{q} \varepsilon(z_j, q)$$

where $\varepsilon(z_j, q)$ represents the elasticity of input z_j (i.e., the percent increase in input z_j as a result of a 1% increase in output q).

- Define total costs as $c = w \cdot z$. Differentiating wrt q, we obtain the marginal cost

$$MC = \frac{\partial c}{\partial q} = \sum_{j=1}^{n} w_j \frac{\partial z_j}{\partial q} = \sum_{j=1}^{n} \frac{w_j z_j}{q} \frac{q}{z_j} \frac{\partial z_j}{\partial q} = \sum_{j=1}^{n} \frac{w_j z_j}{q} \varepsilon(z_j, q).$$

(b) Now show that output level along the average cost function must satisfy

$$1 = \sum_{j=1}^{n} k_j \varepsilon(z_j, q),$$

where $k_j \equiv \frac{w_j z_j}{w \cdot z}$ represents the expenditure share of input z_j in the cost-minimizing input vector of the firm.

- Rewriting the last expression,

$$MC = \sum_{j=1}^{n} \frac{w_j z_j}{q} \varepsilon(z_j, q) = \sum_{j=1}^{n} \underbrace{\frac{w_j z_j}{c}}_{k_j} \underbrace{\frac{c}{q}}_{AC} \underbrace{\frac{q}{z_j} \frac{\partial z_j}{\partial q}}_{\varepsilon(z_j, q)} = AC \sum_{j=1}^{n} k_j \varepsilon(z_j, q).$$

Therefore, since $AC = MC$, the above expression $MC = AC \sum k_j \varepsilon(z_j, q)$ simplifies to

$$1 = \sum_{j=1}^{n} k_j \varepsilon(z_j, q).$$

Thus with $n = 2$, either both elasticities are equal to 1 or one is greater and the other is less than 1. Intuitively, if the firm increases its output by 1%, and if its usage of input 1 increases as a result by more than 1%, then the usage of input 2 must increase by less than 1%.

(c) From the above results, show that in the case that the firm only uses two inputs, 1 and 2, if $\varepsilon(z_2, q) > \varepsilon(z_1, q)$, then

$$\varepsilon(z_2, q) > 1 > \varepsilon(z_1, q).$$

Using the results from part (b), we have that

$$\sum_{j=1}^{n} k_j \varepsilon(z_j, q) = 1$$

thus implying that, for the case of two inputs,

$$k_1 \varepsilon(z_1, q) + \acute{k}_2 \varepsilon(z_2, q) = 1$$

Hence, if $\varepsilon(z_2, q) > \varepsilon(z_1, q)$, it must be that

$$\varepsilon(z_2, q) > 1 > \varepsilon(z_1, q)$$

(d) Let us now consider a parametric example. A firm has production function $f(z) = (z_1 - a)^\alpha z_2^\beta$, where $z_1 \geq a > 0$, and $z_2 \geq 0$. The input price vector is w. Show that along the output expansion path the following equality must hold

$$w_2 z_2 = \frac{\beta}{\alpha} \left(w_1 z_1 - a w_1 \right).$$

Then, show that along the output expansion path $\varepsilon(z_2, q) > 1 > \varepsilon(z_1, q)$ holds.

- The firm solves the following CMP

$$\min_{z_1, z_2} \quad w_1 z_1 + w_2 z_2 \qquad \text{subject to} \quad (z_1 - a)^\alpha z_2^\beta \geq q$$

At the tangency point we find

$$\frac{MP_1}{MP_2} = \frac{\alpha (z_1 - a)^{\alpha - 1} z_2^\beta}{\beta (z_1 - a)^\alpha z_2^{\beta - 1}} = \frac{\alpha z_2}{\beta (z_1 - \alpha)} = \frac{w_1}{w_2}$$

which yields an output expansion path of

$$w_2 z_2 = \frac{\beta}{\alpha}(w_1 z_1 - a w_1)$$

Because $w_2 z_2 = \frac{\beta}{\alpha}(w_1 z_1 - \alpha w_1)$is follows that

$$w_2 z_2 < \frac{\beta}{\alpha} w_1 z_1 \text{ and } w_2 \frac{\partial z_2}{\partial q} = \frac{\beta}{\alpha} w_1 \frac{\partial z_1}{\partial q}.$$

Hence

$$\frac{\beta}{\alpha} \frac{w_1 z_1}{w_2 z_2} > 1 \text{ and } w_2 z_2 \frac{q}{z_2} \frac{\partial z_2}{\partial q} = \frac{\beta}{\alpha} w_1 z_1 \frac{q}{z_1} \frac{\partial z_1}{\partial q}.$$

The last expression can be rewritten as follows:

$$w_2 z_2 \underbrace{\frac{q}{z_2}\frac{\partial z_2}{\partial q}}_{\varepsilon(z_2,q)} = \frac{\alpha}{\beta} w_1 z_1 \underbrace{\frac{q}{z_1}\frac{\partial z_1}{\partial q}}_{\varepsilon(z_1,q)}$$

That is,

$$w_2 z_2 \varepsilon(z_2, q) = \frac{\beta}{\alpha} w_1 z_1 \varepsilon(z_1, q)$$

or,

$$\varepsilon(z_2, q) = \left(\frac{\beta}{\alpha}\frac{w_1 z_1}{w_2 z_2}\right)\varepsilon(z_1, q).$$

Hence $\varepsilon(z_2, q) > \varepsilon(z_1, q)$ because $\left(\frac{\beta}{\alpha}\frac{w_1 z_1}{w_2 z_2}\right) > 1$.

(e) Finally, show that the scale of the active firms will rise if one of the input prices rises, while the scale will fall if the other input price rises.

- In order to answer this question, note that if w_2 increases, both AC and MC will also increase. However, MC rises less than AC, and thus, individual output of the firms in the industry increases, iff $\varepsilon(z_2, q) < 1$. (For a formal proof, see Reley (2012), page 129). In contract, in the exercise $\varepsilon(z_2, q) > 1$. Hence, an increase in w_2 yields a large increase in MC than in AC. Thus at the initial cost-minimizing output, $MC > AC$. Therefore as w_2 rises the average cost-minimizing output rises.

4. **[Cost function that does not increase in output]** Consider a firm with production function

$$q = \frac{bL}{a - K}.$$

where $a, b > 0$, L denotes units of labor, and K represents units of capital. Solving for K in the above production function, we obtain that the isoquant curve is $K = a - \frac{b}{q}L$. An increase in the output level that the firm seeks to produce, q, does not alter the vertical intercept of the isoquant, a, but produces an outward pivoting effect of the isoquant since its slope $\frac{b}{q}$ decreases in q.

(a) Find the marginal rate of technical substitution, $MRTS_{L,K}$.

(b) Set the $MRTS_{L,K}$ equal to the price ratio $\frac{w}{r}$. For which values of labor price, w, we obtain a corner solution where the firm uses capital alone, i.e., $(z_L, z_K) = (0, a)$? [*Hint*: You need to find that w is higher than a certain cutoff, $w > \overline{w}$.]

(c) Assuming that w satisfies the condition you found in part (b), $w > \overline{w}$, find the cost function $c(w, q)$.

(d) Show that the conditional factor demand correspondence $z(w, q)$ found in part (b) and the cost function $c(w, q)$ found in part (c) are both constant in output q.

- This exercise was moved to Homework assignment #5.

5. **Exercise from NS:**

(a) Chapter 11: Exercise 11.12.

- (See answer key at the end of this handout.)

ECONS 501 – MICROECONOMIC THEORY I

HOMEWORK #4 – ANSWER KEY

Exercise from Nicholson and Snyder

11.12 More on derived demand with two inputs. This problem shows how an industry's demand for an input can be computed and why that demand will depend on the elasticity of demand for the good being produced. This is a nice problem therefore for tying together input and output markets.

a. By Shephard's lemma, each partial derivative gives the quantity of input demanded to produce one unit of output. Multiplication by Q gives total industry demand.

b. Under the assumption of constant returns to scale,

$$P = MC = C\left(v, w, 1\right).$$

So in equilibrium,

$$Q = D(MC) = D(C(v, w, 1)).$$

Furthermore,

$$K = QC_v = D(C(v,w,1))C_v$$
$$L = QC_w = D(C(v,w,1))C_w,$$

implying

$$\frac{\partial K}{\partial v} = D'C_v^2 + C_{vv}Q$$
$$\frac{\partial L}{\partial w} = D'C_w^2 + C_{ww}Q.$$

c. Because costs are homogenous of degree 1, the derivatives of C are homogeneous of degree zero. Hence,

$$vC_{vv} + wC_{wv} = 0,$$

implying

$$C_{vv} = \left(\frac{-w}{v}\right)C_{wv}.$$

Similarly, $C_{ww} = \left(\frac{-v}{w}\right)C_{wv}.$

d. We have

$$\sigma = \frac{CC_{wv}}{C_w C_v}.$$

Rearranging, $C_{wv} = \frac{\sigma C_w C_v}{C}$. Replacing C_{wv} in the relation proved in part c yields

$$C_{vv} = \frac{-w}{v} \cdot \frac{\sigma C_w C_v}{C}$$
$$C_{ww} = \frac{-v}{w} \cdot \frac{\sigma C_w C_v}{C}.$$

Replacing C_{vv} and C_{ww} with the above expressions, and C_v and C_w with the expressions from part a, we rewrite the equations from part b as

$$\frac{\partial K}{\partial v} = -\frac{\sigma w K L}{Q v C} + \frac{D' K^2}{Q^2}$$

$$\frac{\partial L}{\partial w} = -\frac{\sigma v K L}{Q w C} + \frac{D' L^2}{Q^2}.$$

e. $$e_{L,w} = \frac{\partial L}{\partial w} \cdot \frac{w}{L} = \frac{-\sigma v K}{QC} + \frac{D' w L P}{Q^2 P} = -\sigma s_K + s_L e_{Q,P}.$$

$$e_{k,v} = \frac{\partial K}{\partial v} \cdot \frac{v}{K} = \frac{-\sigma w L}{QC} + \frac{D' v K P}{Q^2 P} = -\sigma s_L + s_K e_{Q,P}.$$

f. The terms $-\sigma s_L$ and $-\sigma s_K$ are a mathematical representation of the substitution effect. Because the sign of σ is positive and s_L is positive, the overall sign of the substitution effect will be negative. The size of the effect increases when the goods are closer substitutes (when σ is larger) and when we have a larger share of the other input (which makes it more easily replaceable).

The terms $s_K e_{Q,P}$ and $s_L e_{Q,P}$ are the mathematical representation of the output effect. Assuming the demand elasticity is negative, the output effect will be negative and resulting own price elasticity of the inputs will be negative. The output effect will increase the more elastic the demand for the output and the larger the share of the input (a larger share implies that a price increase for the input will have a larger effect on marginal cost and on price).