1. **[Duality in production theory]** Consider a firm with production function \( q = \sqrt{z} \), using one input (e.g., labor) to produce units of output \( q \). The price of every unit of input is \( w > 0 \), and the price of every unit of output is \( p > 0 \).

   (a) Set up the firm’s profit-maximization problem (PMP), and solve for its unconditional factor demand \( z(w, p) \).

   (b) What is the output level that arises from using the amount of inputs \( z(w, p) \)? Label this output level \( q(w) \).

   (c) Set up the firm’s cost-minimization problem (CMP), and solve for its conditional factor demand \( z(w; q) \) for any output level \( q \). (For now, we write the constraint of the CMP to be \( f(z) \geq q \), where the output level \( q \) that the firm seeks to reach does not necessarily coincide with that found in part (b), \( q(w) \).)

   (d) Evaluate the conditional factor demand \( z(w; q) \) at output level \( q = q(w) \), to obtain \( z(w; q(w)) \). Show that it coincides with the unconditional factor demand \( z(w, p) \) found in part (a), that is,

   \[ z(w, q(w)) = z(w, p). \]

   (e) **Shephard’s lemma.** Evaluate the CMP’s objective function, \( w \cdot z \), at the conditional factor demand \( z(w, q) \), to obtain the cost function, that is, find \( c(w, q) = w \cdot z(w, q) \). Differentiate the cost function with respect to \( w \), and show that your result coincides with the conditional factor demand \( z(w, q) \).

   (f) **Substitution and output effects.** Let us now consider that the firm faces cheaper wages (lower \( w \)). Differentiate the unconditional factor demand \( z(w, p) \) found in part (a) with respect to \( w \) to find the total effect of this price change.

   (g) Differentiate the conditional factor demand \( z(w, q) \) found in part (c) with respect to \( w \) to obtain the substitution effect of this price change.

   (h) Compare your results in parts (f) and (g). Which is the output effect of the change in \( w \)?

2. **[Convex cost function].** Consider a firm with Cobb-Douglas production function \( q = f(z_1, z_2) = z_1^\alpha z_2^\beta \), where \( \alpha, \beta > 0 \). Assume that the firm faces input prices \( w_1 \) and \( w_2 \), both being positive.

   (a) Set up the firm’s cost minimization problem (CMP) and find the conditional factor demand correspondence, \( z_i(w, q) \), for every input \( i = \{1, 2\} \).

   (b) Show that the conditional factor demand \( z(w, q) \) satisfies homogeneity of degree zero in \( w \).

   (c) Find the cost function \( c(w, q) = w_1 z_1(w, q) + w_2 z_2(w, q) \).
(d) Show that the cost function \( c(w,q) \) satisfies: (i) homogeneity of degree one in \( w \); (ii) it is weakly increasing in output \( q \); and (iii) \( \frac{dc(w,q)}{dw} = z_i(w,q) \) for every input \( i = \{1, 2\} \) (recall that this last property is commonly referred as Shepard’s lemma).

(e) Show that if the production function \( q = f(z_1, z_2) = z_1^\alpha z_2^\beta \) satisfies constant returns to scale (which occurs when \( \alpha + \beta = 1 \)), then \( z(w,q) \) and \( c(w,q) \) are homogeneous of degree 1 in output \( q \).

(f) What happens with your results in part (e) if the production function exhibits increasing returns to scale \((\alpha + \beta > 1)\)? What if it exhibits decreasing returns to scale \((\alpha + \beta < 1)\)?

(g) Under which conditions is the production function \( q = f(z_1, z_2) = z_1^\alpha z_2^\beta \) concave in input \( z \)? Under which conditions is the cost function \( c(w,q) \) convex in output \( q \)?

3. [Scale of firms in a perfectly competitive market] Consider an industry in which there is free entry, whereby all firms have the same U-shaped AC curve.

(a) Show that for each firm the marginal cost curve is

\[
MC = \sum_{j=1}^{n} \frac{w_jz_j}{q} \varepsilon(z_j, q)
\]

where \( \varepsilon(z_j, q) \) represents the output elasticity (i.e., the percent increase in \( q \) as a result of a 1% increase in input \( z_j \)).

(b) Now show that output level along the average cost function must satisfy

\[
1 = \sum_{j=1}^{n} k_j \varepsilon(z_j, q),
\]

where \( k_j \equiv \frac{w_jz_j}{wz} \) represents the expenditure share of input \( z_j \) in the cost-minimizing input vector of the firm.

(c) From the above results, show that in the case that the firm only uses two inputs, 1 and 2, if \( \varepsilon(z_2, q) > \varepsilon(z_1, q) \), then

\[
\varepsilon(z_2, q) > 1 > \varepsilon(z_1, q).
\]

(d) Let us now consider a parametric example. A firm has production function \( f(z) = (z_1 - a)^\alpha z_2^\beta \), where \( z_1 \geq a > 0 \), and \( z_2 \geq 0 \). The input price vector is \( w \). Show that along the output expansion path the following equality must hold

\[
w_2z_2 = \frac{\beta}{\alpha} (w_1z_1 - aw_1).
\]

Then, show that along the output expansion path \( \varepsilon(z_2, q) > 1 > \varepsilon(z_1, q) \) holds.

(e) Finally, show that the scale of the active firms will rise if one of the input prices rises, while the scale will fall if the other input price rises.
4. [Cost function that does not increase in output] Consider a firm with production function

\[ q = \frac{bL}{a - K}. \]

where \( a, b > 0 \), \( L \) denotes units of labor, and \( K \) represents units of capital. Solving for \( K \) in the above production function, we obtain that the isoquant curve is \( K = a - \frac{b}{q}L \). An increase in the output level that the firm seeks to produce, \( q \), does not alter the vertical intercept of the isoquant, \( a \), but produces an outward pivoting effect of the isoquant since its slope \( \frac{b}{q} \) decreases in \( q \).

(a) Find the marginal rate of technical substitution, \( MRTS_{L,K} \).

(b) Set the \( MRTS_{L,K} \) equal to the price ratio \( \frac{w}{r} \). For which values of labor price, \( w \), we obtain a corner solution where the firm uses capital alone, i.e., \( (z_L, z_K) = (0, a) ? \) \[ Hint: You need to find that \( w \) is higher than a certain cutoff, \( w > \bar{w} \). \]

(c) Assuming that \( w \) satisfies the condition you found in part (b), \( w > \bar{w} \), find the cost function \( c(w, q) \).

(d) Show that the conditional factor demand correspondence \( z(w, q) \) found in part (b) and the cost function \( c(w, q) \) found in part (c) are both constant in output \( q \).

5. Exercise from NS:

(a) Chapter 11: Exercise 11.12.