# EconS 501 - Microeconomic Theory I
# Homework #4 - Due date: October 21st

1. **[Duality in production theory]** Consider a firm with production function $q = \sqrt{z}$, using one input (e.g., labor) to produce units of output $q$. The price of every unit of input is $w > 0$, and the price of every unit of output is $p > 0$.

   (a) Set up the firm's profit-maximization problem (PMP), and solve for its unconditional factor demand $z(w, p)$.

   (b) What is the output level that arises from using the amount of inputs $z(w, p)$? Label this output level $q(w)$.

   (c) Set up the firm's cost-minimization problem (CMP), and solve for its conditional factor demand $z(w, q)$ for any output level $q$. (For now, we write the constraint of the CMP to be $f(z) \geq q$, where the output level $q$ that the firm seeks to reach does not necessarily coincide with that found in part (b), $q(w)$.)

   (d) Evaluate the conditional factor demand $z(w, q)$ at output level $q = q(w)$, to obtain $z(w, q(w))$. Show that it coincides with the unconditional factor demand $z(w, p)$ found in part (a), that is,

   $$z(w, q(w)) = z(w, p).$$

   (e) *Shephard's lemma.* Evaluate the CMP's objective function, $w \cdot z$, at the conditional factor demand $z(w, q)$, to obtain the cost function, that is, find $c(w, q) = w \cdot z(w, q)$. Differentiate the cost function with respect to $w$, and show that your result coincides with the conditional factor demand $z(w, q)$.

   (f) *Substitution and output effects.* Let us now consider that the firm faces cheaper wages (lower $w$). Differentiate the unconditional factor demand $z(w, p)$ found in part (a) with respect to $w$ to find the total effect of this price change.

   (g) Differentiate the conditional factor demand $z(w, q)$ found in part (c) with respect to $w$ to obtain the substitution effect of this price change.

   (h) Compare your results in parts (f) and (g). Which is the output effect of the change in $w$?

2. **[Convex cost function].** Consider a firm with Cobb-Douglas production function $q = f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$, where $\alpha, \beta > 0$. Assume that the firm faces input prices $w_1$ and $w_2$, both being positive.

   (a) Set up the firm's cost minimization problem (CMP) and find the conditional factor demand correspondence, $z_i(w, q)$, for every input $i = \{1, 2\}$.

   (b) Show that the conditional factor demand $z(w, q)$ satisfies homogeneity of degree zero in $w$.

   (c) Find the cost function $c(w, q) = w_1 z_1(w, q) + w_2 z_2(w, q)$.

(d) Show that the cost function $c(w, q)$ satisfies: (i) homogeneity of degree one in $w$; (ii) it is weakly increasing in output $q$; and (iii) $\frac{\partial c(w,q)}{\partial w_i} = z_i(w, q)$ for every input $i = \{1, 2\}$ (recall that this last property is commonly referred as Shepard's lemma).

(e) Show that if the production function $q = f(z_1, z_2) = z_1^\alpha z_2^\beta$ satisfies constant returns to scale (which occurs when $\alpha + \beta = 1$), then $z(w, q)$ and $c(w, q)$ are homogeneous of degree 1 in output $q$.

(f) What happens with your results in part (e) if the production function exhibits increasing returns to scale ($\alpha + \beta > 1$)? What if it exhibits decreasing returns to scale ($\alpha + \beta < 1$)?

(g) Under which conditions is the production function $q = f(z_1, z_2) = z_1^\alpha z_2^\beta$ concave in input $z$? Under which conditions is the cost function $c(w, q)$ convex in output $q$?

3. **[Scale of firms in a perfectly competitive market]** Consider an industry in which there is free entry, whereby all firms have the same U-shaped AC curve.

(a) Show that for each firm the marginal cost curve is

$$MC = \sum_{j=1}^{n} \frac{w_j z_j}{q} \varepsilon(z_j, q)$$

where $\varepsilon(z_j, q)$ represents the output elasticity (i.e., the percent increase in q as a result of a 1% increase in input $z_j$).

(b) Now show that output level along the average cost function must satisfy

$$1 = \sum_{j=1}^{n} k_j \varepsilon(z_j, q),$$

where $k_j \equiv \frac{w_j z_j}{w \cdot z}$ represents the expenditure share of input $z_j$ in the cost-minimizing input vector of the firm.

(c) From the above results, show that in the case that the firm only uses two inputs, 1 and 2, if $\varepsilon(z_2, q) > \varepsilon(z_1, q)$, then

$$\varepsilon(z_2, q) > 1 > \varepsilon(z_1, q).$$

(d) Let us now consider a parametric example. A firm has production function $f(z) = (z_1 - a)^\alpha z_2^\beta$, where $z_1 \geq a > 0$, and $z_2 \geq 0$. The input price vector is $w$. Show that along the output expansion path the following equality must hold

$$w_2 z_2 = \frac{\beta}{\alpha} (w_1 z_1 - a w_1).$$

Then, show that along the output expansion path $\varepsilon(z_2, q) > 1 > \varepsilon(z_1, q)$ holds.

(e) Finally, show that the scale of the active firms will rise if one of the input prices rises, while the scale will fall if the other input price rises.

4. **[Cost function that does not increase in output]** Consider a firm with production function

$$q = \frac{bL}{a - K}.$$

where $a, b > 0$, $L$ denotes units of labor, and $K$ represents units of capital. Solving for $K$ in the above production function, we obtain that the isoquant curve is $K = a - \frac{b}{q}L$. An increase in the output level that the firm seeks to produce, $q$, does not alter the vertical intercept of the isoquant, $a$, but produces an outward pivoting effect of the isoquant since its slope $\frac{b}{q}$ decreases in $q$.

(a) Find the marginal rate of technical substitution, $MRTS_{L,K}$.

(b) Set the $MRTS_{L,K}$ equal to the price ratio $\frac{w}{r}$. For which values of labor price, $w$, we obtain a corner solution where the firm uses capital alone, i.e., $(z_L, z_K) = (0, a)$? [*Hint*: You need to find that $w$ is higher than a certain cutoff, $w > \overline{w}$.]

(c) Assuming that $w$ satisfies the condition you found in part (b), $w > \overline{w}$, find the cost function $c(w, q)$.

(d) Show that the conditional factor demand correspondence $z(w, q)$ found in part (b) and the cost function $c(w, q)$ found in part (c) are both constant in output $q$.

5. **Exercise from NS:**

(a) Chapter 11: Exercise 11.12.