Microeconomic Theory I
Assignment #3 - Answer key

1. [**A utility function generating a linear demand.**] Consider an individual with utility function

$$u(x_0, x_1, x_2) = x_0 + a(x_1 + x_2) - \frac{1}{2}\left(bx_1^2 + 2dx_1x_2 + bx_2^2\right)$$

where x_0 is a composite commodity embodying all goods other than x_1 and x_2, and whose price is normalized to one. The budget constraint is, hence, $x_0 + p_1x_1 + p_2x_2 = w$. We assume that $b > |d|$, which implies that the products are differentiated.

(a) Find the inverse demand functions of good 1 and 2, $p_1(x_0, x_1, x_2)$ and $p_2(x_0, x_1, x_2)$. (For simplicity, you can focus on interior solutions.) Show that they are linear in x_1 and x_2.

- At an interior optimum, the tangency conditions $\frac{MU_1}{MU_0} = \frac{p_1}{p_0}$ and $\frac{MU_2}{MU_0} = \frac{p_2}{p_0}$ must hold. In our setting, $MU_0 = 1$ and $p_0 = \$1$, implying that the tangency conditions reduce to $MU_1 = p_1$ and $MU_2 = p_2$, which entail

$$\begin{aligned} a - bx_1 - dx_2 &= p_1 \quad \text{and} \\ a - bx_2 - dx_1 &= p_2 \end{aligned}$$

which are exactly the indirect demand functions we wanted to find. Both functions are linear in x_1 and x_2.

(b) Solve for x_1 and x_2 in the indirect demand functions found in part (a) to obtain the Walrasian demand function of good 1 and 2, $x_1(p_1, p_2)$ and $x_2(p_1, p_2)$. Show that they are linear in p_1 and p_2.

- Inverting and solving the system of equations in (a) we find that

$$\begin{aligned} x_1(p_1, p_2) &= \frac{a}{b+d} - \frac{b}{(b^2 - d^2)}p_1 + \frac{d}{(b^2 - d^2)}p_2 \quad \text{and} \\ x_2(p_1, p_2) &= \frac{a}{b+d} - \frac{d}{(b^2 - d^2)}p_1 + \frac{b}{(b^2 - d^2)}p_2 \end{aligned}$$

For compactness, let us denote $A \equiv \frac{a}{b+d}$, $B \equiv \frac{b}{(b^2-d^2)}$, and $D \equiv \frac{d}{(b^2-d^2)}$, which yields the Walrasian demands

$$\begin{aligned} x_1(p_1, p_2) &= A - Bp_1 + Dp_2 \quad \text{and} \\ x_2(p_1, p_2) &= A - Dp_1 + Bp_2 \end{aligned}$$

both being linear in p_1 and p_2.

(c) Show that the consumer surplus in this setting is

$$CS(x_0, x_1, x_2) = \frac{1}{2}\left(bx_1^2 + 2dx_1x_2 + bx_2^2\right)$$

- Since demands are linear, we can find the consumer surplus by summing up the area of the triangle below $p_1(x_0, x_1, x_2)$, which gives us CS_1, and the area of the triangle below $p_2(x_0, x_1, x_2)$, which provides us with CS_2. That is,

$$\begin{aligned} CS_1 &= \frac{1}{2}x_1\left[a - (a - bx_1 - dx_2)\right] \\ &= \frac{1}{2}bx_1^2 + dx_1x_2 \end{aligned}$$

and, similarly,

$$\begin{aligned} CS_2 &= \frac{1}{2}x_2\left[a - (a - bx_2 - dx_1)\right] \\ &= \frac{1}{2}bx_2^2 + dx_2x_1 \end{aligned}$$

thus yielding a total consumer surplus of

$$\begin{aligned} CS &= \left(\frac{1}{2}bx_1^2 + dx_1x_2\right) + \left(\frac{1}{2}bx_2^2 + dx_2x_1\right) \\ &= \frac{1}{2}\left(bx_1^2 + 2dx_1x_2 + bx_2^2\right) \end{aligned}$$

(d) Evaluate inverse demand functions found in part (a), and the consumer surplus found in part (c) in the case in which goods 1 and 2 are perfect substitutes, $b = d$. Explain.

- When $b = d$, the inverse demand function of good 1 simplifies to

$$p_1(x_0, x_1, x_2) = a - bx_1 - bx_2 = a - b(x_1 + x_2) = a - bX$$

where $X \equiv x_1 + x_2$ denotes aggregate output. Note that this is the usual expression of the inverse demand when analyzing homogeneous goods. A similar argument applies to the inverse demand function of good 2,

$$p_2(x_0, x_1, x_2) = a - bx_2 - bx_1 = a - bX.$$

- When $b = d$, the consumer surplus found in part (c) becomes

$$\begin{aligned} CS &= \frac{1}{2}\left(bx_1^2 + 2bx_1x_2 + bx_2^2\right) = \frac{1}{2}b\left(x_1^2 + 2x_1x_2 + x_2^2\right) \\ &= \frac{1}{2}b(x_1 + x_2)^2 = \frac{1}{2}bX^2 \end{aligned}$$

where $X \equiv x_1 + x_2$, as defined above. Note that this is the usual expression of consumer surplus when analyzing homogeneous goods with linear demand $p(X) = a - bX$.

2. [**Comprehensive exam, August 2011**] Consider a representative consumer in an economy with J goods, $j = 1, 2, ..., J$. Since we are mainly interested in this individual's consumption of goods 1 and 2, we group all the remaining goods $j = 3, 4, ..., J$ as good

zero, q_0. The price of good zero is normalized to $p_0 = 1$ (i.e., good zero thus becomes the numeraire). The prices of goods 1 and 2 are p_1 and p_2, and income is $m > 0$. This consumer's preferences are represented by utility function

$$u(q_1, q_2, q_0) = q_1^{\frac{1}{4}} q_2^{\frac{1}{4}} + q_0$$

(a) Find the Walrasian demands and the associated indirect utility function.

- **UMP:** In order to solve this problem, we use a standard argument for additively separate utility functions: define $e^R(\mathbf{p}, m) \equiv p_1 q_1^W + p_2 q_2^W$ to be the amount of money spent on purchasing the Walrasian demand of goods 1 and 2 alone. Then, the pair $\left(q_1^W, q_2^W\right)$ must solve the auxiliary problem

$$\max_{q_1, q_2} \; q_1^{\frac{1}{4}} q_2^{\frac{1}{4}} \tag{1}$$

$$\text{subject to} \quad p_1 q_1 + p_2 q_2 = e^R(\mathbf{p}, m)$$

Solving for q_2 in the constraint, $q_2 = \frac{e^R}{p_2} - \frac{p_1}{p_2} q_1$, and plugging it into the objective function, the maximization problem reduces to one with a single choice variable, q_1, as follows

$$\max_{q_1} \; q_1^{\frac{1}{4}} \left(\frac{e^R}{p_2} - \frac{p_1}{p_2} q_1 \right)^{\frac{1}{4}}$$

Taking first order conditions with respect to q_1,

$$\frac{e^R - 2p_1 q_1}{4 p_2 q_1^{\frac{3}{4}} \left(\frac{e^R - p_1 q_1}{p_2} \right)^{\frac{3}{4}}} = 0$$

and solving for q_1 yields

$$q_1^W(\mathbf{p}, e^R) = \frac{e^R}{2p_1}$$

Plugging $q_1^W(\mathbf{p}, e^R) = \frac{e^R}{2p_1}$ into the constraint, $q_2 = \frac{e^R}{p_2} - \frac{p_1}{p_2} q_1$, we obtain

$$q_2^W(\mathbf{p}, e^R) = \frac{e^R}{2p_2}$$

In addition, note that $q_1^W(\mathbf{p}, e^R)$ and $q_2^W(\mathbf{p}, e^R)$ do not depend on the overall income of the individual, m, but on the amount of income he spends on good 1 and 2 alone, e^R. Expressions $q_1^W(\mathbf{p}, e^R)$ and $q_2^W(\mathbf{p}, e^R)$ yield an associated utility level of

$$v^R(\mathbf{p}, e^R) = \left(\frac{1}{p_1} \right)^{1/4} \left(\frac{1}{p_2} \right)^{1/4} \left(\frac{e^R}{2} \right)^{1/2}$$

which can be interpreted as the indirect utility function of the auxiliary maximization problem (1).

- Given these results for goods 1 and 2, we can analyze good 0. In particular, the Walrasian demand for good 0, q_0^W, and the amount of income spent on goods 1 and 2, $e^R(\mathbf{p}, m)$, must solve

$$\max_{q_0, e^R} \ v^R(\mathbf{p}, e^R) + q_0$$

$$\text{subject to} \ \ e^R(\mathbf{p}, m) + q_0 = m$$

Furthermore, since $q_0 = m-\ e^R(\mathbf{p}, m)$, the above program can be simplified to the following maximization problem (with only one choice variable):

$$\max_{e^R} \ g(e^R, \mathbf{p}) = v^R(\mathbf{p}, e^R) + \left[m - e^R(p, m)\right]$$

Taking first order conditions with respect to e^R, we obtain

$$\frac{\partial g(e^R, \mathbf{p})}{\partial e^R} = \frac{\left(\frac{1}{p_1}\right)^{1/4}\left(\frac{1}{p_2}\right)^{1/4}}{2\sqrt{2}\sqrt{e^R}} - 1 \tag{2}$$

and second order conditions

$$\frac{\partial^2 g(e^R, \mathbf{p})}{\partial e^{R\ 2}} = -\frac{\left(\frac{1}{p_1}\right)^{1/4}\left(\frac{1}{p_2}\right)^{1/4}}{4\sqrt{2}\left(e^R\right)^{3/2}} < 0 \tag{3}$$

showing that the objective function $g(e^R, \mathbf{p})$ is strictly concave.

- Therefore, from the first order conditions in (2), the value of $e^R(\mathbf{p}, m)$ that maximizes $g(e^R, \mathbf{p})$ is $e^*(\mathbf{p}) = \frac{1}{8\sqrt{p_1 p_2}}$. This implies that:
  - When $m > \frac{1}{8\sqrt{p_1 p_2}}$, Walrasian demands are

$$q_1^W = \frac{\frac{1}{8\sqrt{p_1 p_2}}}{2p_1} = \frac{1}{16\sqrt{p_1^3 p_2}} \quad \text{and} \quad q_2^W = \frac{\frac{1}{8\sqrt{p_1 p_2}}}{2p_2} = \frac{1}{16\sqrt{p_1 p_2^3}}$$

    for goods 1 and 2, and the rest of income, $q_0^W = m - \frac{1}{8\sqrt{p_1 p_2}}$, is spent on good 0. (Interior solutions).
  - By contrast, when $m \leq \frac{1}{8\sqrt{p_1 p_2}}$, no income is spent on good 0, $q_0^W = 0$, but only on goods 1 and 2, that is

$$q_1^W = \frac{m}{2p_1} \quad \text{and} \quad q_2^W = \frac{m}{2p_2}$$

    at a corner solution.
- Hence, the Walrasian demand correspondence can be summarized as

$$\left(q_1^W, q_2^W, q_0^W\right) = \begin{cases} \left(\frac{1}{16\sqrt{p_1^3 p_2}}, \frac{1}{16\sqrt{p_1 p_2^3}}, m - \frac{1}{8\sqrt{p_1 p_2}}\right) & \text{if } m > \frac{1}{8\sqrt{p_1 p_2}}, \text{ and} \\ \left(\frac{m}{2p_1}, \frac{m}{2p_2}, 0\right) & \text{if } m \leq \frac{1}{8\sqrt{p_1 p_2}}. \end{cases}$$

Note that, at the interior solution, the Walrasian demands of goods 1 and 2 do not depend on income, implying that these goods do not exhibit income effects, since all additional income effect is entirely spent on the numeraire good.

- From the above Walrasian demands, it is easy to obtain the associated indirect utility function

$$v(\mathbf{p}, m) = \begin{cases} m + \frac{1}{8\sqrt{p_1 p_2}} & \text{if } m > \frac{1}{8\sqrt{p_1 p_2}}, \text{ and} \\ \left(\frac{m^2}{4p_1 p_2}\right)^{1/4} & \text{if } m \leq \frac{1}{8\sqrt{p_1 p_2}}. \end{cases}$$

(b) Invert the indirect utility function $v(\mathbf{p}, m)$ to obtain the expenditure function $e(\mathbf{p}, u)$.

- Note that in order to obtain the expenditure function $e(\mathbf{p}, u)$, we just need to invert the indirect utility function $v(\mathbf{p}, m)$, i.e., solving for m, which yields

$$e(\mathbf{p}, u) = \begin{cases} u - \frac{1}{8\sqrt{p_1 p_2}} & \text{if } u > \frac{1}{4\sqrt{p_1 p_2}}, \text{ and} \\ 2u^2\sqrt{p_1 p_2} & \text{if } u \leq \frac{1}{4\sqrt{p_1 p_2}}. \end{cases}$$

(c) Consider that the price vector increases from $\mathbf{p^0} = (p_1^0, p_2^0) = (1, 1)$ to $\mathbf{p}^1 = (p_1^1, p_2^1) = (2, 1)$, i.e., only the price of good 1 doubles. Let us next use the equivalent variation (EV) to evaluate the welfare loss that the consumer suffers from the increase in the price of good 1. In order to keep track of the possible corner solutions that arise at different income levels, we separately evaluate the EV at different values of m.

1. What is the EV when income satisfies $m > \frac{1}{8}$, i.e., the consumer is relatively rich?

   - In this case, the consumer is at the interior solution both *before* and *after* the price change. In particular,

   $$u^0 = v(\mathbf{p}^0, m) = m + \frac{1}{8} \text{ and } u^1 = v(\mathbf{p}^1, m) = m + \frac{1}{8\sqrt{2}}$$

   and the corresponding expenditure functions are

   $$e(\mathbf{p}^0, u^0) = u^0 - \frac{1}{8} = m \text{ and } e(\mathbf{p}^1, u^1) = u^1 - \frac{1}{8\sqrt{2}} = m$$

   and

   $$e(\mathbf{p}^0, u^1) = u^1 - \frac{1}{8} = m + \frac{1}{8\sqrt{2}} - \frac{1}{8} \text{ and } e(\mathbf{p}^1, u^0) = u^0 - \frac{1}{8\sqrt{2}} = m + \frac{1}{8} - \frac{1}{8\sqrt{2}}$$

   - Therefore, the equivalent variation (EV) is

   $$EV = e(\mathbf{p}^0, u^0) - e(\mathbf{p}^0, u^1) = m - \left(m + \frac{1}{8\sqrt{2}} - \frac{1}{8}\right) \simeq 0.036$$

   where note that we define the EV as the negative of the standard definition, since in this case we measure a loss in consumer welfare. Intuitively, the EV measures the additional income that we need to give to this consumer after the price increase, for him to maintain the same utility level he reached before the price increase.

2. What is the EV when income satisfies $\frac{1}{8} > m > \frac{1}{8\sqrt{2}}$, i.e., the consumer is moderately rich?
   - *Utility levels.* In this case, the initial equilibrium *before* the price change is at a corner solution, while the equilibrium *after* the price change is interior. In particular,

$$u^0 = v(\mathbf{p}^0, m) = \left(\frac{m^2}{4}\right)^{1/4} \quad \text{and } u^1 = v(\mathbf{p}^1, m) = m + \frac{1}{8\sqrt{2}}$$

   - *Expenditure function* $e(\mathbf{p}^0, u^0)$. The expenditure functions that we need to use in each case depend on whether the utility level we are using (u^0 or u^1) exceed the cutoff $\frac{1}{4\sqrt{p_1 p_2}}$, as we described in the previous part of the exercise when we found the piecewise expenditure function $e(p, u)$. In particular, for utility level $u^0 = \left(\frac{m^2}{4}\right)^{1/4}$, we have that $u^0 \leq \frac{1}{4\sqrt{p_1 p_2}}$ since $\left(\frac{m^2}{4}\right)^{1/4} < \frac{1}{4}$ holds given that $m < \frac{1}{8}$. Hence, for utility level u^0 we need to use expenditure function $2u^2\sqrt{p_1 p_2}$, as follows

$$e(\mathbf{p}^0, u^0) = 2\left[\left(\frac{m^2}{4}\right)^{1/4}\right]^2 = 2\sqrt{\frac{m^2}{4}} = m$$

   - *Expenditure function* $e(\mathbf{p}^1, u^1)$. For the case of utility level u^1 we have that $u^1 > \frac{1}{4\sqrt{p_1 p_2}}$ holds given that $m + \frac{1}{8\sqrt{2}} > \frac{1}{4\sqrt{2}}$ is satisfied for all $m > \frac{1}{8\sqrt{2}}$. Since this part of the exercise assumes that m satisfies $\frac{1}{8} > m > \frac{1}{8\sqrt{2}}$, we have that we need to use $u - \frac{1}{8\sqrt{p_1 p_2}}$ as the expenditure function. In particular,

$$e(\mathbf{p}^1, u^1) = \underbrace{\left(m + \frac{1}{8\sqrt{2}}\right)}_{u^1} - \frac{1}{8\sqrt{2}} = m$$

   - *Expenditure function* $e(\mathbf{p}^0, u^1)$. Similarly, in order to find expenditure function $e(\mathbf{p}^0, u^1)$, notice that for utility level u^1 we have that $u^1 > \frac{1}{4\sqrt{p_1 p_2}}$ holds (as discussed above). Hence, we need to use $u - \frac{1}{8\sqrt{p_1 p_2}}$ as the expenditure function. Importantly, note that our testing of whether u^1 exceeds cutoff $\frac{1}{4\sqrt{p_1 p_2}}$ must always be evaluated at the prices at which u^1 is evaluated ($\mathbf{p}^1$ price vector), regardless of the prices at which we afterwards seek to evaluate the expenditure function. In particular, for expenditure function $e(\mathbf{p}^0, u^1)$, which is evaluated at the original price vector $\mathbf{p}^0$,.we have

$$e(\mathbf{p}^0, u^1) = u_1 - \frac{1}{8\sqrt{1}} = \underbrace{\left(m + \frac{1}{8\sqrt{2}}\right)}_{u^1} - \frac{1}{8}$$

   - Therefore, the equivalent variation (EV) is

$$EV = e(\mathbf{p}^0, u^0) - e(\mathbf{p}^0, u^1) = m - \left(m + \frac{1}{8\sqrt{2}} - \frac{1}{8}\right) = \frac{1}{8} - \frac{1}{8\sqrt{2}} \simeq 0.036$$

3. What is the EV when income satisfies $\frac{1}{8\sqrt{2}} > m$, i.e., the consumer is poor?
   - In this case, the equilibrium is at a corner solution, both *before* and *after* the price change. In particular,

$$u^0 = v(\mathbf{p}^0, m) = \left(\frac{m^2}{4}\right)^{1/4} \quad \text{and } u^1 = v(\mathbf{p}^1, m) = \left(\frac{m^2}{8}\right)^{1/4}$$

     and the corresponding expenditure functions are

$$e(\mathbf{p}^0, u^0) = 2\sqrt{\frac{m^2}{4}} = m \quad \text{and} \quad e(\mathbf{p}^1, u^1) = 2\sqrt{2\frac{m^2}{8}} = m$$

     and

$$e(\mathbf{p}^0, u^1) = 2\sqrt{\frac{m^2}{8}} = \frac{m}{\sqrt{2}}$$

   - Therefore, the equivalent variation (EV) is

$$EV = e(\mathbf{p}^0, u^0) - e(\mathbf{p}^0, u^1) = m - \frac{m}{\sqrt{2}} = \left(1 - \frac{1}{\sqrt{2}}\right) m \simeq 0.29m$$

3. **[Utility maximization in France.]** Bernard's preferences are defined over three commodities: wine, bread, and leisure. Yes, he is French! Let x_1 denote his consumption of wine, x_2 be his consumption of bread, and x_3 be his consumption of leisure. Assume that his consumption set is the vector space $\mathbb{R}_+ \times \mathbb{R}_+ \times [0, H]$, meaning that Bernard can consume any positive amount of wine and bread but he cannot consume more than H hours of leisure (e.g., H could represent the total number of hours in a day). Let $H > 0$, and suppose that his preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = \frac{x_1^{\alpha} x_2^{1-\alpha}}{\alpha^{\alpha} (1-\alpha)^{1-\alpha}} + \ln x_3$$

   He faces prices p_1 and p_2 for wine and bread, respectively. Last, Bernard is endowed with H hours of leisure and M dollars of non-wage income, and he can work L hours at a wage of w dollars per hour.

   (a) If Bernard were to spend E dollars on wine and bread, show that the optimal way for him to allocate his expenditure of E dollars between wine and bread is independent of his consumption of leisure. Show that the maximum 'utility' he can generate by the expenditure of E dollars on wine and bread is given by the indirect utility function

$$v(p_1, p_2, E) = \frac{E}{p_1^{\alpha} p_2^{1-\alpha}}$$

   (b) Set $X = v(p_1, p_2, E)$ to be the (real) consumption of Bernard and set $p = p_1^{\alpha} p_2^{1-\alpha}$, to be the 'price' of (real) consumption. Explain why the vector (X^*, L^*), Bernard's optimal amount of consumption and his optimal supply of labor, it the solution to the following constrained maximization problem

$$\max_{(X,L)} X + \ln(H - L)$$

$$\text{subject to} \;\; pX \leq M + wL, \;\; X \geq 0, \text{ and } \;\; L \in [0, H].$$

(c) Assuming an interior solution to the constrained maximization problem in part (b), derive as functions of the prices (p, w) and Bernard's non-wage income M, first X (his consumption) and second L his supply of labor.

(d) Define the function $m(p, w, u)$ to be the minimum non-wage income Bernard requires to achieve the utility u when facing a 'price of consumption' p and wage rate w. That is,

$$m(p, w, u) = \min_{(X,L)} pX - wL \quad \text{subject to } X + \ln(H - L) \geq u$$

Show that

$$m(p, w, u) = (1 + u - \ln p + \ln w)\, p - wH$$

[*Hint*: Do not solve the constrained minimization problem directly. Think what $v(p, w, m(p, w, u))$ must be identically equal to and use that identity along with your answer to part (c).]

(e) Differentiate $m(p, w, u)$ with respect to p and w, respectively. To what do $\frac{\partial m(p,w,u)}{\partial p}$ and $\frac{\partial m(p,w,u)}{\partial w}$ correspond?

(f) Suppose in the initial situation $m^0 = 0$, $p = p^0$ and $w = w^0$, and there are no taxes on consumption or labor. Now suppose in the new situation the government introduces a tax on labor at the uniform rate $\tau \in (0, 1)$, so the net wage Bernard receives for every hour of labor becomes $w^1 = (1 - \tau)w^0$. What happens to the quantity of labor he supplies and the amount of goods he consumes? What is the deadweight loss associated with this tax on labor?

- See answer key at the end of this handout.

EconS 501 – Homework #3

Answer key

Exercise #3. Bernard's preferences are defined over three commodities: wine, bread, and leisure. Let x_1 denote his consumption of wine, x_2 his consumption of bread, x_3 his consumption of leisure. Suppose his consumption set is the vector space $\mathbb{R}_+ \times \mathbb{R}_+ \times [0, H]$, meaning that Bernard can consume any positive amount of wine and bread but he cannot consume more than *H* hours of leisure. Let $0 < H$, and suppose his preferences can be represented by the utility function:

$$U(x_1, x_2, x_3) = \frac{x_1^{\alpha} x_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}} + \ln x_3$$

Bernard is endowed with *H* hours of leisure and *M* dollars of non-wage income. He can work as many hours *L* as he wants at *W* dollars per hour and he faces linear prices p_1 and p_2 for wine and bread, respectively.

(a) If Bernard were to spend *E* dollars on wine and bread, show that the optimal way for him to allocate his expenditure of *E* dollars between wine and bread is independent of his consumption of leisure. Show that the maximum 'utility' he can generate by the expenditure of *E* dollars on wine and bread is given by the function:

$$v(p_1, p_2, E) = \frac{E}{p_1^{\alpha} p_2^{1-\alpha}}$$

ANSWER:

This problem can be seen to be a standard UMP with Cobb-Douglas preferences:

$$\max_{(x_1,x_2)} u(x_1, x_2) = \frac{x_1^{\alpha} x_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}$$
$$s.t. \quad p_1 x_1 + p_2 x_2 \le E$$

Form the Lagrangian and obtain first order optimality conditions (FONCs)

$$
\begin{aligned}
&L = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - E) \\
&\frac{\partial L}{\partial x_1} = \alpha \frac{u(x^*_1, x^*_2)}{x^*_1} - \lambda^* p_1 \le 0, \quad \text{with equality if } x^*_1 > 0 \\
&\frac{\partial L}{\partial x_2} = (1-\alpha) \frac{u(x^*_1, x^*_2)}{x^*_2} - \lambda^* p_2 \le 0, \quad \text{with equality if } x^*_2 > 0 \\
&\frac{\partial L}{\partial \lambda} = p_1 x^*_1 + p_2 x^*_2 - E \le 0, \quad \text{with equality if } \lambda^* > 0
\end{aligned}
$$

For interior solution, first two FONCs yield:

$$\left(\frac{\alpha}{1-\alpha}\right) p_2 x_2^* = p_1 x_1^*$$

Substituting into $\frac{\partial L}{\partial \lambda}$ (the expenditure constraint):

$$\left(1+\frac{\alpha}{1-\alpha}\right) p_2 x_2^* = E$$

$$\Rightarrow x_2^* = \frac{(1-\alpha)E}{p_2}$$

We also find:

$$x_1^* = \frac{\alpha E}{p_1}$$

Plugging into the utility function $u(x_1, x_2)$ yields:

$$
\begin{aligned}
v(p_1, p_2, E) &= \frac{\left(\frac{\alpha E}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)E}{p_2}\right)^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}} \\
&= \frac{E}{p_1^{\alpha} p_2^{1-\alpha}}, \qquad \text{as required}
\end{aligned}
$$

(b) Set $X = v(p_1, p_2, E)$ to be the (real) consumption of Bernard and set $P = p_1^{\alpha} p_2^{1-\alpha}$, to be the 'price' of (real) consumption. Explain why (X^*, L^*), Bernard's optimal amount of consumption and his optimal supply of labor, is the solution to the following constrained maximization problem:

$$\max_{(X,L)} X + \ln(H - L)$$
$$s.t. \quad PX \leq M + WL$$
$$X \geq 0$$
$$L \in [0, H]$$

ANSWER:

Notice that $H - L = x_3$ and using our result from part (a) we have:

$$X = v(p_1, p_2, E) = \max_{(x_1, x_2)} u(x_1, x_2) \quad s.t. \; p_1x_1 + p_2x_2 \leq E$$

Next we note that $U(x_1, x_2, x_3)$ is additively separable between (x_1, x_2), and x_3. That is:

$$U(x_1, x_2, x_3) = u(x_1, x_2) + \ln x_3$$

We know that for weakly separable preferences, the UMP

$$\max_{(x_1, x_2, x_3)} U(x_1, x_2, x_3) \qquad s.t. \; p_1x_1 + p_2x_2 + Wx_3 \leq M + WH$$

is equivalent to the UMP:

$$\max_{(X,L)} v(p_1, p_2, E) + \ln(H - L - h) \qquad s.t. \; E = PX \leq M + WL$$

And we can obtain $x_3^* = H - L^*$, and using Roy's identity (MWG Prop 3.G.4 found on page 73):

$$x_1^* = \frac{\dfrac{\partial v(p_1, p_2, E^*)}{\partial p_1}}{\dfrac{\partial v(p_1, p_2, E)}{\partial w}} = \frac{\alpha E^*}{p_1} = \frac{\alpha PX^*}{p_1}$$

and

$$x_2^* = \frac{\dfrac{\partial v(p_1, p_2, E^*)}{\partial p_2}}{\dfrac{\partial v(p_1, p_2, E)}{\partial w}} = \frac{(1-\alpha)E^*}{p_2} = \frac{(1-\alpha)PX^*}{p_2}$$

(c) Assuming an interior solution to the constrained maximization problem in part (b), derive as functions of the prices (*P*, *W*) and Bernard's non-wage income *M*, first *X* (his consumption) and second *L* his supply of labor.

ANSWER:

Form the Lagrangian

$$\mathcal{L} = X + \ln(H - L) - \Lambda(PX - M - WL)$$

which yields the following FONCs:

$$\frac{\partial \mathcal{L}}{\partial E} = 1 - \Lambda^* P = 0, \qquad \text{(because we are assuming } X^* > 0\text{)}$$

$$\frac{\partial \mathcal{L}}{\partial L} = -\frac{1}{(H - L)} + \Lambda^* W = 0 \qquad \text{(because we are assuming L}^* > 0\text{)}$$

$$\frac{\partial \mathcal{L}}{\partial \Lambda} = PX^* \text{ - } M \text{ - } WL^* = 0 \qquad \text{(because we are assuming } \Lambda^* > 0\text{)}$$

From the first two FONCs we obtain:

$$L(P, W, M) = H - \frac{P}{W}$$

Plugging this result into the third FONC (the expenditure constraint) yields:

$$X(P, W, M) = \frac{M + WH}{P} - 1$$

(d) Define the function $m(P,W,U)$ to be the minimum non-wage income Bernard requires to achieve the utility U when facing a 'price of consumption' P and wage rate W. That is:

$$m(P,W,U) = \min_{(X,L)} PX - WL \qquad s.t. \;\; X + \ln(H - L) \geq U$$

Show that:

$$m(P,W,U) = (U + 1 - \ln P + \ln W)P - WH$$

[*Hint*: Do not solve the constrained minimization problem directly. Think what $V(P,W,m(P,W,U))$ must be identically equal to and use that identity along with your answer to part (c).]

ANSWER:

Using the answer from part (c) we have:

$$V(P,W,M) = X(P,W,M) + \ln(H - L(P,W,M))$$
$$= \frac{M + WH}{P} - 1 + \ln P - \ln W$$

This is the maximum utility the individual can achieve when he has non-wage income of *M*, and faces a price of consumption *P* and wage rate *W*. Now by definition:

$$V(P,W,m(P,W,U)) = U$$

That is, if Bernard has non-wage income of $m(P,W,U)$ and faces a price of consumption P and a wage rate W, then the maximum utility he can achieve must be U. So:

$$\frac{m(P,W,U)+WH}{P}-1+\ln P-\ln W=U$$

Hence:

$$m(P,W,U)=(U+1-\ln P+\ln W)P-WH, \qquad \text{as required}$$

(e) Differentiate $m(P,W,U)$ with respect to P and W, respectively. To what do $\frac{\partial m(P,W,U)}{\partial P}$ and $\frac{\partial m(P,W,U)}{\partial W}$ correspond?

ANSWER:

By the envelope theorem:

$$\frac{\partial m(P,W,U)}{\partial P}=X(P,W,U)=U-\ln P+\ln W$$
$$\frac{\partial m(P,W,U)}{\partial W}=-L(P,W,U)=\frac{P}{W}-H=-L(P,W,M)$$

Notice that the compensated and uncompensated labor supplies coincide for Bernard (this is because the non-wage income effects are zero).

(f) Suppose in the initial situation $M^0=0, P=P^0$ and $W=W^0$ and there are no taxes on consumption or labor. Now suppose in the new situation the government introduces a tax on labor at the uniform rate $\tau\in(0,1)$, so the net wage Bernard receives for an hour of labor becomes $W^1=(1-\tau)W^0$. What happens to the quantity of labor he supplies to the market and the amount he consumes? What is the deadweight loss associated with this tax on labor?

ANSWER:

The tax leads him to supply less labor and to consume less (real) consumption. Using our results from part c) of the exercise, the fall in the amount of labor he supplies is given by:

$$L(P^0,W^0,0)-L(P^0,(1-\tau)W^0,0)=\frac{P^0}{W^0}\left[\frac{1}{1-\tau}-1\right]$$
$$=\frac{P^0}{W^0}\times\frac{\tau}{(1-\tau)}$$

while the fall in his (real) consumption is

$$X(P^0, W^0, 0) - X(P^0, (1-\tau)W^0, 0) = \frac{\tau W^0 H}{P^0}$$

Following lectures, a measure for the deadweight loss is given by:

$$DWL = -EV - T$$

This situation is slightly different from the one we considered in class, where the individual was choosing how to spend a given amount of wealth among the various different commodities at given fixed prices. With the standard UMP problem it made sense to evaluate the *EV* by:

$$EV = e(p^0, u^1) - e(p^0, u^0)$$

But our consumer's utility maximization problem was to select the optimal amount of consumption *X* to demand and labor *L* to supply subject to 'budget constraint':

$$PX = M + WL$$

In our current setting it makes sense to use a money-metric indirect utility function based on the function $m(P, W, U)$, defined and derived in part (c). We take the equivalent variation of the tax to be: the change in non-wage income that induces the same change in utility as that induced by the tax. In symbols that definition is:

$$\begin{aligned} EV &= m(P^0, W^0, U^1) - m(P^0, W^0, U^0) \\ &= m(P^0, W^0, U^1) - 0 \\ &= (U^1 + 1 - \ln P^0 + \ln W^0)P^0 - W^0 H \end{aligned}$$

We have:

$$\begin{aligned} U^1 &= X(P^0, (1-\tau)W^0, 0) + \ln(H - L(P^0, (1-\tau)W^0, 0)) \\ &= \frac{(1-\tau)W^0 H}{P^0} - 1 + \ln P^0 - \ln W^0 - P^0 \ln(1-\tau) \end{aligned}$$

We combine and simplify to,

$$EV = -\tau W^0 H + P^0 \ln(1-\tau)$$

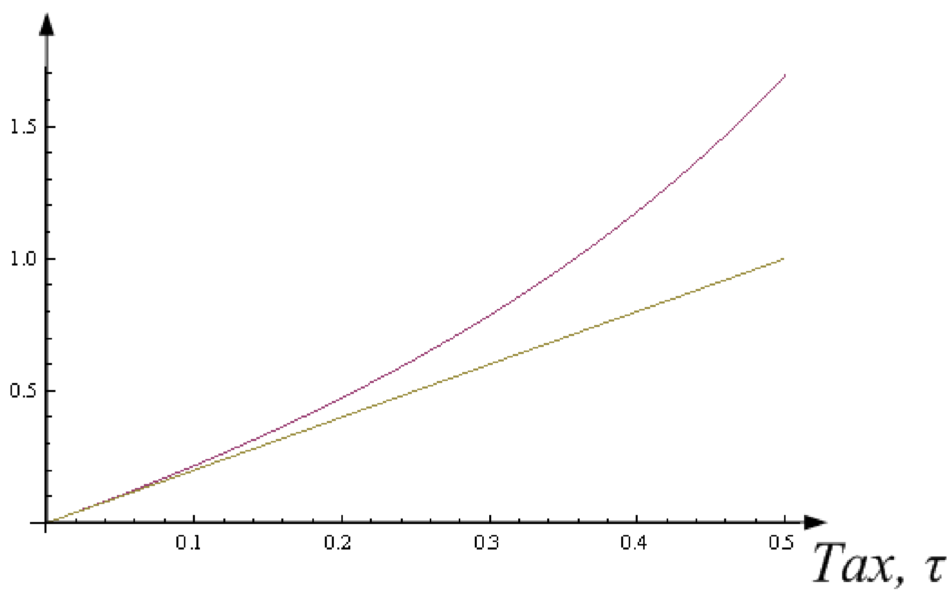
And the tax revenue raised is:

$$TAX = \tau W^0 \times L(P^0, (1-\tau)W^0, 0) = \tau \left(W^0 H - \frac{P^0}{(1-\tau)} \right)$$

Thus,

$$DWL = -EV - TAX$$
$$= \tau W^0 H - P^0 \ln(1-\tau) - \tau\left(W^0 H - \frac{P^0}{(1-\tau)}\right)$$
$$= P^0\left[\frac{\tau}{1-\tau} + \ln\left(\frac{1}{1-\tau}\right)\right]$$

Finally, note that the term in brackets is very close to 2τ when the tax τ is relatively small. For illustrative purposes, the next figure depicts $\frac{\tau}{1-\tau} + \ln\left(\frac{1}{1-\tau}\right)$ as a function of the tax (a convex function), and 2τ (the linear function), confirming that both curves are extremely close for taxes below 10% (below 0.1 in the horizontal axis of the figure).



thus implying that the *DWL* can be closely approximated by

$$DWL \approx 2\tau P^0$$