Microeconomic Theory I
Assignment #3 - Due date: September, 26th, in class.

1. [**A utility function generating a linear demand.**] Consider an individual with utility function

$$u(x_0, x_1, x_2) = x_0 + a(x_1 + x_2) - \frac{1}{2}\left(bx_1^2 + 2dx_1x_2 + bx_2^2\right)$$

where x_0 is a composite commodity embodying all goods other than x_1 and x_2, and whose price is normalized to one. The budget constraint is, hence, $x_0 + p_1x_1 + p_2x_2 = w$. We assume that $b > |d|$, which implies that the products are differentiated.

   (a) Find the inverse demand functions of good 1 and 2, $p_1(x_0, x_1, x_2)$ and $p_2(x_0, x_1, x_2)$. (For simplicity, you can focus on interior solutions.) Show that they are linear in x_1 and x_2.

   (b) Solve for x_1 and x_2 in the inverse demand functions found in part (a) to obtain the Walrasian demand function of good 1 and 2, $x_1(p_1, p_2)$ and $x_2(p_1, p_2)$. Show that they are linear in p_1 and p_2.

   (c) Show that the consumer surplus in this setting is

$$CS(x_0, x_1, x_2) = \frac{1}{2}\left(bx_1^2 + 2dx_1x_2 + bx_2^2\right)$$

   (d) Evaluate inverse demand functions found in part (a), and the consumer surplus found in part (c) in the case in which goods 1 and 2 are perfect substitutes, $b = d$. Explain.

2. [**Comprehensive exam, August 2011**] Consider a representative consumer in an economy with J goods, $j = 1, 2, ..., J$. Since we are mainly interested in this individual's consumption of goods 1 and 2, we group all the remaining goods $j = 3, 4, ..., J$ as good zero. The price of good zero is $p_0 = 1$ (the numeraire). The prices of goods 1 and 2 are p_1 and p_2, and income is $m > 0$. This consumer's preferences are represented by utility function

$$u(q_1, q_2, q_0) = q_1^{\frac{1}{4}} q_2^{\frac{1}{4}} + q_0$$

   (a) Find the Walrasian demands and the associated indirect utility function. Invert the indirect utility function to obtain the expenditure function.

   (b) Consider that the price vector increases from $\mathbf{p^0} = (p_1^0, p_2^0) = (1, 1)$ to $\mathbf{p}^1 = (p_1^1, p_2^1) = (2, 1)$, i.e., only the price of good 1 doubles. Let us next use the equivalent variation (EV) to evaluate the loss in welfare that the consumer suffers from the increase in the price of good 1.

      1. What is the EV when income satisfies $m > \frac{1}{8}$, i.e., the consumer is relatively rich?

2. What is the EV when income satisfies $\frac{1}{8} > m > \frac{1}{8\sqrt{2}}$, i.e., the consumer is moderately rich?
3. What is the EV when income satisfies $\frac{1}{8\sqrt{2}} > m$, i.e., the consumer is poor?

3. **[Utility maximization in France.]** Bernard's preferences are defined over three commodities: wine, bread, and leisure. Yes, he is French! Let x_1 denote his consumption of wine, x_2 be his consumption of bread, and x_3 be his consumption of leisure. Assume that his consumption set is the vector space $\mathbb{R}_+ \times \mathbb{R}_+ \times [0, H]$, meaning that Bernard can consume any positive amount of wine and bread but he cannot consume more than H hours of leisure (e.g., H could represent the total number of hours in a day). Let $H > 0$, and suppose that his preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = \frac{x_1^{\alpha} x_2^{1-\alpha}}{\alpha^{\alpha} (1-\alpha)^{1-\alpha}} + \ln x_3$$

He faces prices p_1 and p_2 for wine and bread, respectively. Last, Bernard is endowed with H hours of leisure and M dollars of non-wage income, and he can work L hours at a wage of w dollars per hour.

(a) If Bernard were to spend E dollars on wine and bread, show that the optimal way for him to allocate his expenditure of E dollars between wine and bread is independent of his consumption of leisure. Show that the maximum 'utility' he can generate by the expenditure of E dollars on wine and bread is given by the indirect utility function

$$v(p_1, p_2, E) = \frac{E}{p_1^{\alpha} p_2^{1-\alpha}}$$

(b) Set $X = v(p_1, p_2, E)$ to be the (real) consumption of Bernard and set $p = p_1^{\alpha} p_2^{1-\alpha}$, to be the 'price' of (real) consumption. Explain why the vector (X^*, L^*), Bernard's optimal amount of consumption and his optimal supply of labor, it the solution to the following constrained maximization problem

$$\max_{(X,L)} X + \ln(H - L)$$

$$\text{subject to } \; pX \leq M + wL, \;\; X \geq 0, \text{ and } \;\; L \in [0, H] \, .$$

(c) Assuming an interior solution to the constrained maximization problem in part (b), derive as functions of the prices (p, w) and Bernard's non-wage income M, first X (his consumption) and second L his supply of labor.

(d) Define the function $m(p, w, u)$ to be the minimum non-wage income Bernard requires to achieve the utility u when facing a 'price of consumption' p and wage rate w. That is,

$$m(p, w, u) = \min_{(X,L)} pX - wL \quad \text{subject to } X + \ln(H - L) \geq u$$

Show that

$$m(p, w, u) = (1 + u - \ln p + \ln w)\, p - wH$$

[*Hint*: Do not solve the constrained minimization problem directly. Think what $v(p, w, m(p, w, u))$ must be identically equal to and use that identity along with your answer to part (c).]

(e) Differentiate $m(p, w, u)$ with respect to p and w, respectively. To what do $\frac{\partial m(p,w,u)}{\partial p}$ and $\frac{\partial m(p,w,u)}{\partial w}$ correspond?

(f) Suppose in the initial situation $m^0 = 0$, $p = p^0$ and $w = w^0$, and there are no taxes on consumption or labor. Now suppose in the new situation the government introduces a tax on labor at the uniform rate $\tau \in (0, 1)$, so the net wage Bernard receives for every hour of labor becomes $w^1 = (1 - \tau)w^0$. What happens to the quantity of labor he supplies and the amount of goods he consumes? What is the deadweight loss associated with this tax on labor?