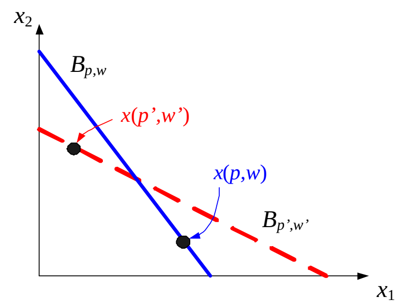
Microeconomic Theory I
Assignment #2 - Answer Key

1. **[Checking WARP].** For each of the following demand functions, check whether they satisfy the weak axiom of revealed preference (WARP).

   (a) "Random demand": For any pair of prices p_1 and p_2 and wealth w, the consumer randomizes uniformly over all points in the budget frontier.

   - Let us prove it by a counterexample.



Random demand

WARP states that

$$\text{if } p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w) \text{ then } p' \cdot x(p, w) > w'$$

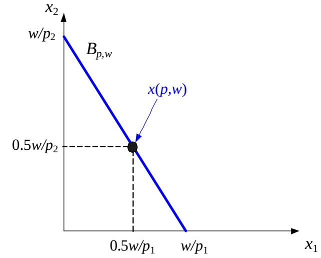
That is, if the new consumption bundle (chosen under the new prices and wealth) would not have been affordable under the old prices and wealth, then it must be the case that the old consumption bundle is not affordable under the new prices and wealth. In this case, we find that

$$p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w) \textbf{ but } p' \cdot x(p, w) < w'$$

That is, the old consumption bundle is *still* affordable under the new prices and wealth. Hence, there exists a probability greater than zero that random demand assigns bundles as the ones chosen in the figure. As a consequence we can conclude that random demand does not satisfy WARP.

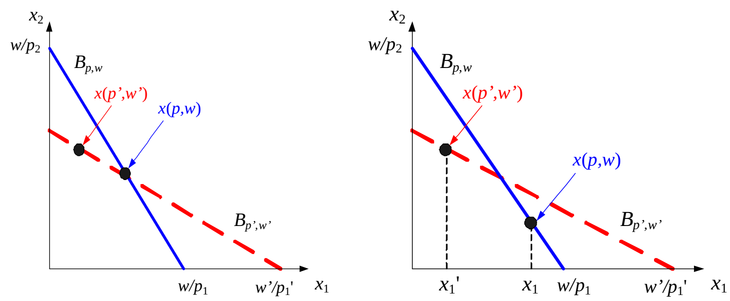
   (b) "Average demand": The expected "random demand" given p_1, p_2 and w.

   - First, note that if the consumer randomizes uniformly over all points in her budget line, then the expected random demand is allocated at the midpoint of the budget line.



Average demand

Let us now prove that WARP is satisfied for average demand. Let us work by contradiction, by assuming that average demand violates WARP. There are two possibilities in which this violation might take place, as the following two figures illustrate.



- Let us compute point x_1 and x_1'. Recall that these points have to be allocated at the midpoint of the budget line. Hence,

$$x_1 = \frac{1}{2}\frac{w}{p_1} \quad \text{and } x_1' = \frac{1}{2}\frac{w'}{p_1'}$$

therefore $2x_1 = \frac{w}{p_1}$ and $2x_1' = \frac{w'}{p_1'}$. Moreover, we can see in both figures that $x_1' < x_1$. Therefore, $2x_1' < 2x_1$, which implies

$$\frac{w'}{p_1'} < \frac{w}{p_1}$$

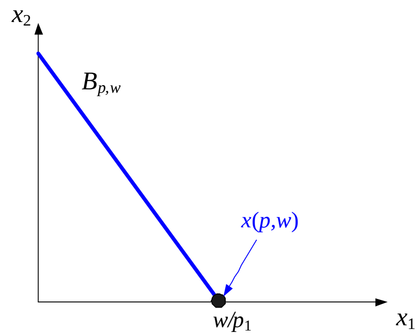
But in both figures we in fact see that $\frac{w'}{p_1'} > \frac{w}{p_1}$. Hence, we have reached a contradiction, and average demand cannot violate WARP.

(c) "Conspicuous demand": For any p_1, p_2 and w,

$$x_i(p_1, p_2, w) = \begin{cases} \frac{w}{p_i} & \text{if } \frac{w}{p_i} = \min\left\{\frac{w}{p_1}, \frac{w}{p_2}\right\} \text{ and } \frac{w}{p_1} \neq \frac{w}{p_2} \\ \frac{w}{p_i} & \text{if } \frac{w}{p_1} = \frac{w}{p_2} \text{ and } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

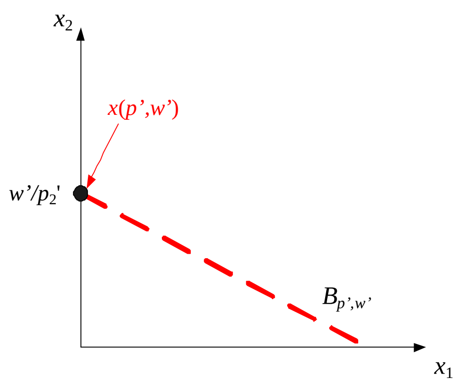
- Let us divide it into two cases: Case 1, in which $\frac{w}{p_1} \leq \frac{w}{p_2}$, and Case 2, in which $\frac{w}{p_1} > \frac{w}{p_2}$.
- *Case 1* $\left(\frac{w}{p_1} \leq \frac{w}{p_2}\right)$: The demand function to the following expression

$$x_i(p_1, p_2, w) = \begin{cases} \frac{w}{p_1} & \text{if } \frac{w}{p_1} \neq \frac{w}{p_2} \\ \frac{w}{p_1} & \text{if } \frac{w}{p_1} = \frac{w}{p_2} \\ 0 & \text{otherwise} \end{cases}$$

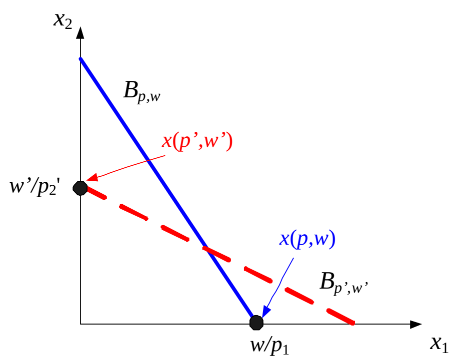


- *Case 2* $\left(\frac{w}{p_1} > \frac{w}{p_2}\right)$: The demand function to the following expression

$$x_i\left(p_1, p_2, w\right) = \begin{cases} \frac{w}{p_2} & \text{if } \frac{w}{p_1} \neq \frac{w}{p_2} \\ 0 & \text{otherwise} \end{cases}$$



- Finally, in this figure we represent both cases simultaneously.



- Let us now check if conscopious demand satisfies WARP. As mentioned in part (a), WARP states that

  $$\text{if } p \cdot x\left(p', w'\right) \leq w \text{ and } x\left(p', w'\right) \neq x\left(p, w\right) \text{ then } p' \cdot x\left(p, w\right) > w'$$

  But as we can see in the figure that deals with both cases, we have that $p' \cdot x\left(p, w\right) < w'$. Hence, conscopious demand violates WARP.

2. **[Stone-Geary utility function].** Exercise 4.12 from Nicholson and Snyder.

- See answer key at the end of the handout.

3. **[UMP, EMP, and Duality].** A consumer has a Cobb-Douglas utility function

$$u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha},$$

where $\alpha > 0$ and $x_1, x_2 \in \mathbb{R}_+$. Assume that the price vector satisifes $p \equiv (p_1, p_2) >> 0$, and wealth $w > 0$.

(a) Write the consumer's utility maximization problem. Find Walrasian demands, and the indirect utility function function $v(p_1, p_2, w)$.

- The consumer's UMP is

$$\max_{x_1, x_2} \; x_1^{\alpha} x_2^{1-\alpha} \text{s.t.} \;\; p_1 x_1 + p_2 x_2 \leq w.$$

And the Lagrangian is

$$L = x_1^{\alpha} x_2^{1-\alpha} + \lambda (w - p_1 x_1 + p_2 x_2)$$

Taking first-order conditions, we obtain

$$\alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 = 0 \Longleftrightarrow \frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{p_1} = \lambda$$

and

$$(1-\alpha) x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 = 0 \Longleftrightarrow \frac{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}}{p_2} = \lambda$$

The above two conditions imply that

$$\frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{p_1} = \frac{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}}{p_2}$$

or rearranging,

$$p_1 x_1 = \frac{\alpha}{1-\alpha} p_2 x_2$$

From the budget constraint, $p_1 x_1 + p_2 x_2 = w$, or $p_2 x_2 = w - p_1 x_1$, which we can insert in the right-hand side of the above expression to obtain

$$p_1 x_1 = \frac{\alpha}{1-\alpha} (w - p_1 x_1).$$

Solving for x_1, we find the Walrasian demand for good 1,

$$x_1(p, w) = \frac{\alpha w}{p_1},$$

- Using the budget constraint again, $p_2 x_2 = w - p_1 x_1$ or $x_2 = \frac{w - p_1 x_1}{p_2}$ we can insert the Walrasian demand for good 1 in order to obtain that of good 2,

$$\begin{aligned} x_2(p, w) &= \frac{w - p_1 \frac{\alpha w}{p_1}}{p_2} \\ &= \frac{(1-\alpha) w}{p_2}. \end{aligned}$$

Then inserting Walrasian demands into the utility function $x_1^{\alpha} x_2^{1-\alpha}$, we find the indirect utility function

$$\begin{aligned} v(p_1, p_2, w) &= \left(\frac{\alpha w}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha) w}{p_2}\right)^{1-\alpha} \\ &= \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} w. \end{aligned}$$

(b) Write the consumer's expenditure minimization problem. Find Hicksian demands, and the expenditure function $e(p_1, p_2, u)$.

- The EMP is

$$\min_{x_1, x_2} \quad p_1 x_1 + p_2 x_2 \text{s.t.} \quad u(x_1, x_2) \equiv x_1^{\alpha} x_2^{1-\alpha} \geq u.$$

The Lagrangian in this case is

$$L = p_1 x_1 + p_2 x_2 - \mu \left(x_1^{\alpha} x_2^{1-\alpha} - u\right)$$

Taking the first-order conditions, we have

$$p_1 - \mu \alpha x_1^{\alpha-1} x_2^{1-\alpha} = 0 \Longleftrightarrow \frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{p_1} = \frac{1}{\mu}$$

and

$$p_2 - \mu (1-\alpha) x_1^{\alpha} x_2^{-\alpha} = 0 \Longleftrightarrow \frac{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}}{p_2} = \frac{1}{\mu}$$

The above two conditions imply that

$$\frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{p_1} = \frac{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}}{p_2}$$

or rearranging,

$$x_1 = \frac{\alpha}{1-\alpha} \frac{p_2}{p_1} x_2$$

- Substituting it into the constraint $x_1^{\alpha} x_2^{1-\alpha} = u$, we obtain

$$\left(\frac{\alpha}{1-\alpha} \frac{p_2}{p_1} x_2\right)^{\alpha} x_2^{1-\alpha} = u$$

which simplifies to

$$\left(\frac{\alpha}{1-\alpha} \frac{p_2}{p_1}\right)^{\alpha} x_2^{\alpha+(1-\alpha)} = u$$

which solving for x_2, yields the Hicksian demand for good 2,

$$h_2(p, u) = \left[\frac{(1-\alpha) p_1}{\alpha p_2}\right]^{\alpha} u.$$

Therefore, we can use $x_1 = \frac{\alpha}{1-\alpha}\frac{p_2}{p_1}x_2$ to find the Hicksian demand for good 1, as follows

$$
\begin{aligned}
h_1(p,u) &= \frac{\alpha}{1-\alpha}\frac{p_2}{p_1}\left[\frac{(1-\alpha)p_1}{\alpha p_2}\right]^{\alpha} u \\
&= \left[\frac{\alpha p_2}{(1-\alpha)p_1}\right]^{1-\alpha} u
\end{aligned}
$$

- Finally, inserting Hicksian demands into the EMP, we get the expenditure function

$$
\begin{aligned}
e(p_1,p_2,u) &= p_1\left[\frac{\alpha p_2}{(1-\alpha)p_1}\right]^{1-\alpha} u + p_2\left[\frac{(1-\alpha)p_1}{\alpha p_2}\right]^{\alpha} u \\
&= \left[\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\right] p_1^{\alpha} p_2^{1-\alpha} u.
\end{aligned}
$$

(c) Evaluate the Walrasian demands $x(p_1,p_2,w)$ at $w = e(p_1,p_2,u)$, and show that Walrasian and Hicksian demands coincide, that is,

$$x(p_1,p_2,e(p_1,p_2,u)) = h(p_1,p_2,u).$$

- The Walrasian demands $x(p_1,p_2,w)$ is the best bundle that maximize the consumer's utility when his/her wealth equals the minimum expenditure. Substituting $w = e(p_1,p_2,u)$ into the Walrasian demand functions, we obtain

$$
\begin{aligned}
x_1(p_1,p_2,e(p_1,p_2,u)) &= \frac{\alpha}{p_1}\left[\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\right] p_1^{\alpha} p_2^{1-\alpha} u \\
&= \left[\frac{\alpha p_2}{(1-\alpha)p_1}\right]^{1-\alpha} u = h_1(p_1,p_2,u)
\end{aligned}
$$

and

$$
\begin{aligned}
x_2(p_1,p_2,e(p_1,p_2,u)) &= \frac{1-\alpha}{p_2}\left[\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\right] p_1^{\alpha} p_2^{1-\alpha} u \\
&= \left[\frac{(1-\alpha)p_1}{\alpha p_2}\right]^{\alpha} u = h_2(p_1,p_2,u).
\end{aligned}
$$

Therefore, Walrasian and Hicksian demands coincide when income equals the minimum expenditure.

(d) Evaluate the Hicksian demands $h(p_1,p_2,u)$ at $u = v(p_1,p_2,w)$, and show that Hicksian and Walrasian demands coincide, that is,

$$h(p_1,p_2,v(p_1,p_2,w)) = x(p_1,p_2,u).$$

- The Hicksian demands is the commodity bundle which make the consumer spend the least to reach utility level u. Substituting $u = v(p_1,p_2,w)$ into the Hicksian demand functions, we obtain

$$
\begin{aligned}
h_1(p_1,p_2,v(p_1,p_2,w)) &= \left[\frac{\alpha p_2}{(1-\alpha)p_1}\right]^{1-\alpha}\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} w \\
&= \frac{\alpha w}{p_1} = x_1(p_1,p_2,,u)
\end{aligned}
$$

and

$$\begin{aligned} h_2\left(p_1, p_2, v\left(p_1, p_2, w\right)\right) &= \left[\frac{(1-\alpha) p_1}{\alpha p_2}\right]^\alpha \left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} w \\ &= \frac{(1-\alpha) w}{p_w} = x_2\left(p_1, p_2, , u\right). \end{aligned}$$

This implies that Hicksian and Walrasian demands coincide at $u = v\left(p_1, p_2, w\right)$.

(e) Evaluate the indirect utility function $v(p_1, p_2, w)$ at $w = e(p_1, p_2, u)$, and show that

$$v(p_1, p_2, e(p_1, p_2, u)) = u.$$

- The indirect utility function $v(p_1, p_2, e\left(p_1, p_2, u\right))$ is the highest utility level the consumer can get when his/her wealth is the minimal expenditure $e\left(p_1, p_2, u\right)$. Inserting $w = e\left(p_1, p_2, u\right)$ into the indirect utility function, we obtain

$$\begin{aligned} v(p_1, p_2, e\left(p_1, p_2, u\right)) &= \left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} \left[\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\right] p_1^\alpha p_2^{1-\alpha} u \\ &= u. \end{aligned}$$

(f) Evaluate the expenditure function $e(p_1, p_2, u)$ at $u = v(p_1, p_2, w)$, and show that

$$e(p_1, p_2, v(p_1, p_2, w)) = w.$$

- The expenditure function $e(p_1, p_2, v(p_1, p_2, w))$ is the minimum wealth that the consumer has to spend in order to reach the utility level $u = v(p_1, p_2, w)$. Substituting $u = v(p_1, p_2, w)$ into the expenditure function, we find

$$\begin{aligned} e(p_1, p_2, v(p_1, p_2, w)) &= \left[\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\right] p_1^\alpha p_2^{1-\alpha} \left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} w \\ &= w. \end{aligned}$$

4. **[Choosing between two types of labor.]** Consider an individual who is endowed with one unit of time which can be allocated to leisure (l) or to either of two types of labor. Generally, the person uses its labor income to purchase consumption goods c at the price p. The first type of labor, L_1, pays a lower wage (w_1) but is easier to perform, whereas the second type, L_2, pays a higher wage ($w_2 > w_1$) but is more difficult. The person's preferences are given by the strictly concave utility function

$$u(c, l, L_1, L_2)$$

where the partial derivatives are $u_c > 0$, $u_l > 0$, and $u_{L_i} < 0$ for both types of labor $i = \{1, 2\}$. In terms of these partial derivatives, one way to formalize the notion that "L_1 is easier than L_2" by wrting that, when the individual works the same amount of time in both types of labor, $L_1 = L_2$, he has $u_{L_1} > u_{L_2}$ implying that, since $u_{L_i} < 0$, the marginal disutility from L_2 is larger (i.e., more negative) than that from L_1.

(a) Assume the person can do L_1 or L_2 but not both, and that it is only possible to choose $L_i \in \left\{0, \frac{1}{3}\right\}$; that is, labor can only be supplied in the discrete amounts 0 or 1/3. Explain how the individual would choose which job, if either, to perform.

- There are three possibilities:
  - $L_1 = L_2 = 0$, which implies that all his time is allocated to leisure, $l = 1$, and that he does not obtain any wage to buy consumption goods, i.e., $c = 0$;
  - $L_1 = \frac{1}{3}$ and $L_2 = 0$, thus implying that the remaining time for leisure is only $l = \frac{2}{3}$, and that he obtains a salary $w_1 \frac{1}{3}$ which allows him to afford $\frac{w_1}{3p}$ units of the consumption good c; or
  - $L_1 = 0$ and $L_2 = \frac{1}{3}$ which also implies a reamining leisure of $l = \frac{2}{3}$, and that he obtains a salary $w_2 \frac{1}{3}$ which allows him to afford $\frac{w_2}{3p}$ units of the consumption good c. (Note that the option $L_1 = \frac{1}{3}$, $L_2 = \frac{1}{3}$ and $l = \frac{1}{3}$ is not allowed in this context, since the individual can only supply one type of labor, but not both.)
- Finally, the individual chooses option 1, 2 or 3, depending on which one yields the highest utility level, i.e., he compares $u(0, 1, 0, 0)$, $u(\frac{w_1}{3p}, \frac{2}{3}, \frac{1}{3}, 0)$ and $u(\frac{w_2}{3p}, \frac{2}{3}, 0, \frac{1}{3})$.

(b) Now assume that labor can be supplied continuously but that again it is not possible to do both jobs. In this case, explain how the individual would choose which job to accept.

- In this case, we first need to solve the value of L_1 that maximizes this individual's utility function, that is,

$$v_1 = \max_{L_1} u\left(\frac{w_1 L_1}{p}, 1 - L_1, L_1, 0\right) \tag{1}$$

  where $L_1 > 0$, $L_2 = 0$, and the remaining leisure time is thus $l = 1 - L_1$. Similarly, if the individual only works in the second job, he chooses L_2 to maximize

$$v_2 = \max_{L_2} u\left(\frac{w_2 L_2}{p}, 1 - L_2, 0, L_2\right) \tag{2}$$

  Once we have the value functions v_1 and v_2, the individual compares v_1 and v_2: if $v_1 > v_2$, then he chooses the argmax of problem (1); while if $v_1 < v_2$ he chooses the argmax of problem (2).

(c) Next, assume it is possible to supply both types of labor. Formulate the individual's decision problem and characterize the solution in terms of the appropriate first order conditions.

- The individual now chooses the values of L_1 and L_2 that solve

$$\max_{L_1, L_2} \; u\left(\frac{w_1 L_1 + w_2 L_2}{p}, 1 - L_1 - L_2, L_1, L_2\right)$$

  where $w_1 L_1 + w_2 L_2$ describes the total salary from both types of labor, and $l = 1 - L_1 - L_2$ denotes the remaining leisure (out of a total time of 1 unit) that this individual can enjoy after working L_1 hours on the first type of job and L_2 hours on the second type of job.

- Taking first order conditions with respect to L_1 we obtain

$$\frac{w_1}{p}u_c - u_l + u_{L_1} \leq 0$$

and similarly with respect to L_2,

$$\frac{w_2}{p}u_c - u_l + u_{L_2} \leq 0$$

(d) Prove that in order for the agent to supply positive quantities of both types of labor, it is necessary that L_2 should be more difficult than L_1 according to the definition of "L_1 is easier than L_2" we described at the beginning of the exercise.

- For an interior solution in which the individual supplies positive amounts of both types of labor, we need that both of the above first order conditions hold with equality, that is

$$\frac{w_1}{p}u_c - u_l + u_{L_1} = 0 \quad \text{and} \quad \frac{w_2}{p}u_c - u_l + u_{L_2} = 0$$

Hence, solving for the common term u_l on both expressions, we obtain

$$\frac{w_1}{p}u_c + u_{L_1} = \frac{w_2}{p}u_c + u_{L_2}$$

and rearranging,

$$\frac{w_1 - w_2}{p}u_c = u_{L_2} - u_{L_1}$$

Therefore, if $w_2 > w_1$, we need that $u_{L_1} > u_{L_2}$ for this expression to hold with equality.

(e) Finally, suppose that while L_1 is easier than L_2 (i.e., entails less effort) it is also more boring. As presently formulated, is the above model sufficient to allow these two factors (effort and boring) to be taken into consideration? If so, explain. Otherwise, discuss how the model might be extended to incorporate them.

- According to the present formulation, all that matters is the combined effect of the two factors (effort and boring), as captured by u_{L_2}. While both factors can play a role, only their combined effect is reflected in the values of the variables of the model. The separate effects cannot be distinguished. Intuitively, on the basis of effort, $u_{L_1} > u_{L_2}$ since "L_2 is more difficult than L_1," but on the basis of boredom, $u_{L_1} < u_{L_2}$ since "L_1 is more tedious than L_2" The current formulation combines the two effects.
- In order to represent the effects separately, it would be necessary to distinguish the two factors in the utility function, such as

$$u(c, l, \underbrace{f(e_1, b_1)}_{L_1}, \underbrace{f(e_2, b_2)}_{L_2}),$$

where e_i denotes the effort in labor type $i = \{1, 2\}$, while b_i represents how boring labor i is.

Answer key – Homework #2

Nicholson and Snyder 4.12, Stone-Geary Utility

a. For $x < x_0$ utility is negative, so he will spend $p_x x_0$ first. With $I- p_x x_0$ extra income, this is a standard Cobb-Douglas problem:

$$p_x(x - x_0) = \alpha (I - p_x x_0), \quad p_y y = \beta(I - p_x x_0)$$

b. Calculating budget shares from part (a) yields

$$\frac{p_x x}{I} = \alpha + \frac{(1-\alpha) p_x x_0}{I} \quad \text{and} \quad \frac{p_y y}{I} = \beta - \frac{\beta p_x x_0}{I}$$

and the limiting points of these two ratios (budget shares) when income approaches infinity are

$$\lim(I \to \infty) \frac{p_x x}{I} = \alpha \quad \text{and} \quad \lim(I \to \infty) \frac{p_y y}{I} = \beta$$