EconS 501 - Micro Theory I
Assignment #1 - Answer key

1. **"Liking more of everything"** Consider a consumption set $X = \mathbb{R}^L$, and define that, for every two bundles $x, y \in X$,

$$x \succsim y \text{ if and only if } x_k \geq y_k \text{ for every component } k,$$

that is, bundle x is at least as good as bundle y if the former contains more units than the latter in each of its components. Check if this preference relation satisfies (a) completeness, (b) transitivity, (c) strong monotonicity, and (d) strict convexity.

   (a) *It is not complete.* Recall that completeness requires for every pair x and y, either $x \succsim y$ or $y \succsim x$ (or both). To see why this property does not hold, consider two bundles $x, y \in \mathbb{R}^2$ with bundle x containing more units of good 1 than bundle y but fewer units of good 2, i.e., $x_1 > y_1$ for good 1 but $x_2 < y_2$ for good 2. Then, we have neither that $x \succsim y$ (since for that we would need $x_1 > y_1$ and $x_2 > y_2$) nor $y \succsim x$ (i.e., for that we would need $y_1 > x_1$ and $y_2 > x_2$).

   (b) *It is transitive.* Recall that transitivity requires that, for any three bundles x, y and z, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$. Now $x \succsim y$ and $y \succsim z$ means that

$$x \geq y \text{ and } y \geq z$$

   In vector notation, this means that bundle x is weakly larger than y in every component. Similarly, bundle y is weakly larger than bundle z in every component. Hence,

$$x \geq y \text{ and } y \geq z \Longrightarrow x_l \geq y_l \text{ and } y_l \geq z_l \text{ for every good } l.$$

   By transitivity of the "greater or equal than" symbol $\geq$ we thus have $x_l \geq z_l$ for all goods l, and so $x \geq z$ (i.e., bundle x is weakly larger than bundle z in every component). That is $x \succsim z$ holds, as required.

   (c) *It is strongly monotone.* Recall that the property of strong monotonicity requires that if we increase one of the goods in a given bundle, then the newly created bundle must be strictly preferred to the original bundle. More compactly, for a given bundle y, if bundle x satisfies

$$x \geq y \text{ and } x \neq y \text{ then } x \succ y$$

   where recall that $x \geq y$ denotes that bundle x contains weakly larger amounts than bundle y in every component. Now $x \geq y$, and $x \neq y$ implies that $x_l \geq y_l$ for all goods l and $x_k > y_k$, for at least one good k (otherwise the two bundles would be the same). Thus $x \geq y$ and $x \neq y$ implies

$$x \succsim y \text{ and } \textit{not } y \succsim x.$$

In words, bundle x being weakly larger than y in all components (but being strictly larger in at least one component, since they cannot completely coincide) implies that bundle x is weakly preferred to y, and it can never be that bundle y is weakly preferred to bundle x. We can therefore conclude that x is strictly preferred to y, $x \succ y$, as required.

(d) *It is strictly convex.* Recall that strict convexity requires if $x \succsim z$ and $y \succsim z$ and $x \neq z$ then the linear combination of bundles x and y, $\alpha x + (1-\alpha)y$, is strictly preferred to z,

$$\alpha x + (1-\alpha)y \succ z \text{ for all } \alpha \in (0,1).$$

Now if $x \succsim z$ and $y \succsim z$ then in the preference relation we are analyzing, it means that bundle x is weakly larger than bundle z in every component, and similarly for bundle y, i.e.,

$$x_l \geq z_l \text{ and } y_l \geq z_l \text{ for all goods } l.$$

And if bundles x and z are different, $x \neq z$, then for some good $k \in \{1, ..., L\}$ we must have $x_k > z_k$. Thus, for any $\alpha \in (0,1)$, the linear combination of x and y lies above bundle z, i.e.,

$$\alpha x_l + (1-\alpha)y_l \geq z_l \text{ for all good } l, \text{ and}$$

$$\alpha x_k + (1-\alpha)y_k > z_k \text{ for some good } k.$$

Hence, we have that $\alpha x + (1-\alpha)y \geq z$ and $\alpha x + (1-\alpha)y \neq z$, and so,

$$\alpha x + (1-\alpha)y \succsim z \text{ and } \textit{not } z \succsim \alpha x + (1-\alpha)y.$$

Therefore $\alpha x + (1-\alpha)y \succ z$, as required for strict convexity.

2. **Checking properties of a preference relation.** Consider a consumer with the following preference relation: he weakly prefers (x_1, x_2) to (y_1, y_2), i.e., $(x_1, x_2) \succsim (y_1, y_2)$, if and only if $\max\{x_1, x_2\} \geqslant \min\{y_1, y_2\}$.

(a) Provide a verbal description of his preference relation.

- Intuitively, this relation states that a bundle (x_1, x_2) is preferred to an alternative bundle (y_1, y_2) if and only if the most abundant component of the first bundle exceeds the least abundant component of the second bundle.

(b) Check whether this preference relation is rational (complete and transitive), monotone, convex, and locally nonsatiated.

1. *Completeness.* For any two bundles (x_1, x_2),$(y_1, y_2) \in \mathbb{R}^2$, either $\max\{x_1, x_2\} \geqslant \min\{y_1, y_2\}$, or $\min\{y_1, y_2\} > \max\{x_1, x_2\}$, or both (note that this occurs when $\max\{x_1, x_2\} = \min\{y_1, y_2\}$, implying that the amount of the most abundant component of bundle x exactly coincides with the amount in the least abundant component in bundle y.). In the first case, we clearly have $(x_1, x_2) \succsim (y_1, y_2)$. In the second, note that $\min\{y_1, y_2\} > \max\{x_1, x_2\}$ implies

$$\max\{y_1, y_2\} \geqslant \min\{y_1, y_2\} > \max\{x_1, x_2\} \geqslant \min\{x_1, x_2\}$$

and hence, $\max\{y_1, y_2\} \geqslant \min\{x_1, x_2\}$, which implies $(y_1, y_2) \succsim (x_1, x_2)$. Hence, this preference relation is complete.

2. *Transitivity.* Take three bundles (x_1, x_2), (y_1, y_2) and $(z_1, z_2) \in \mathbb{R}^2$ satisfying $(x_1, x_2) \succsim (y_1, y_2)$ and $(y_1, y_2) \succsim (z_1, z_2)$. Then, they must satisfy that, on one hand, $\max\{x_1, x_2\} \geqslant \min\{y_1, y_2\}$, and on the other hand, $\max\{y_1, y_2\} \geqslant \min\{z_1, z_2\}$. However, it can be the $\max\{x_1, x_2\} \geqslant \min\{z_1, z_2\}$ is not satisfied, a condition we need for $(x_1, x_2) \succsim (z_1, z_2)$ and thus for transitivity to hold. For instance, consider bundle $(x_1, x_2) = (1, 0)$ and $(y_1, y_2) = (3, 0)$, which satisfy

$$\max\{x_1, x_2\} = 1,\ \ 3 = \min\{y_1, y_2\};\ \text{ and } (z_1, z_2) = (2, 2),$$

which satisfies

$$\max\{y_1, y_2\} = 3 \geqslant 2 = \min\{z_1, z_2\}.$$

However, note that

$$\max\{x_1, x_2\} = 1 < 2 = \min\{z_1, z_2\},$$

which implies $(x_1, x_2) \precsim (z_1, z_2)$. As a consequence, while $(x_1, x_2) \succsim (y_1, y_2)$ and $(y_1, y_2) \succsim (z_1, z_2)$, we cannot conclude that $(x_1, x_2) \succsim (z_1, z_2)$, Entailing that the preference relation is *not* transitive, and thus not rational either.

3. *Monotonicity.* Take a bundle (y_1, y_2), and now let us consider another bundle (x_1, x_2) that contains larger amounts of both goods, i.e., with $x_1 > y_1$ and $x_2 > y_2$. At this point, when comparing the $\max\{x_1, x_2\}$ against the $\min\{y_1, y_2\}$, we can find that $\max\{x_1, x_2\} \geqslant \min\{y_1, y_2\}$ and hence $(x_1, x_2) \succsim (y_1, y_2)$. However, we can also have that $\max\{y_1, y_2\} \geqslant \min\{x_1, x_2\}$, which implies $(y_1, y_2) \succsim (x_1, x_2)$. In order to see that, let us consider the following example: $(x_1, x_2) = (3, 1)$ and $(y_1, y_2) = (2, 0)$. Indeed, note that

$$\max\{x_1, x_2\} = 3 \geqslant \min\{y_1, y_2\} = 0,$$

but also that

$$\max\{y_1, y_2\} = 2 \geqslant \min\{x_1, x_2\} = 1.$$

Hence, this preference relation doesn't satisfy monotonicity. Since, for monotonicity to hold, we need $(x_1, x_2) \succsim (y_1, y_2)$ but $(y_1, y_2) \not\succsim (x_1, x_2)$, so that the bundle in which all components have been increased, (x_1, x_2), is *strictly* preferred to the initial bundle, i.e., $(x_1, x_2) \succ (y_1, y_2)$.

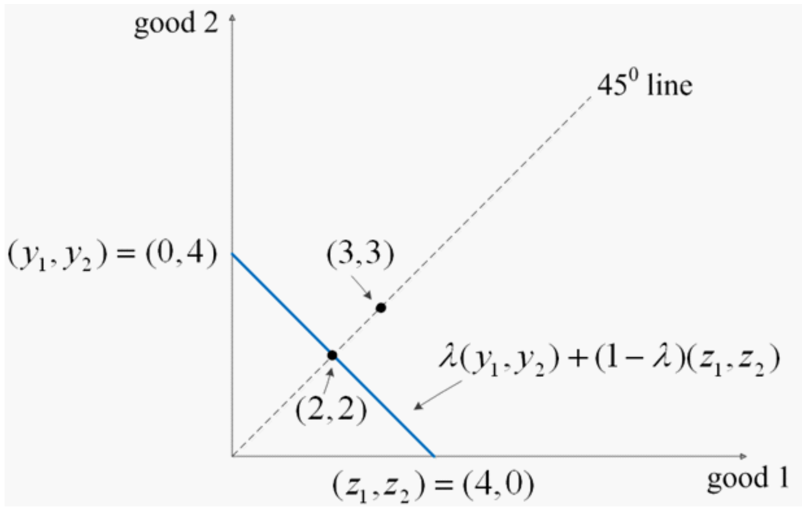
4. *Convexity.* Take three bundles (x_1, x_2),(y_1, y_2) and $(z_1, z_2) \in \mathbb{R}^2$ with $(y_1, y_2) \succsim (x_1, x_2)$ and $(z_1, z_2) \succsim (x_1, x_2)$. Therefore, it must be that $\max\{y_1, y_2\} \geqslant \min\{x_1, x_2\}$, and similarly the $\max\{z_1, z_2\} \geqslant \min\{x_1, x_2\}$. However, the convex combination of (y_1, y_2) and (z_1, z_2) yields,

$$\max\{\lambda y_1 + (1 - \lambda) z_1, \lambda y_2 + (1 - \lambda) z_2\},$$

which is not necessarily higher than $\min\{x_1, x_2\}$. In order to see that, consider an example in which

$$\max\{y_1, y_2\} \geqslant \min\{x_1, x_2\},\ \text{ and }\ \max\{z_1, z_2\} \geqslant \min\{x_1, x_2\},$$

such as $(y_1, y_2) = (0, 4)$, $(x_1, x_2) = (3, 3)$ and $(z_1, z_2) = (4, 0)$. Indeed, $\max\{y_1, y_2\} = 4 \geqslant \min\{x_1, x_2\} = 3$. Now, note that the convex combination of $(y_1, y_2) = (0, 4)$ and $(z_1, z_2) = (4, 0)$ with λ, will give us values between 0 and 4. Graphically, since we examine commodity bundles in $\mathbb{R}^2$, (0, 4) lies on the vertical axis while (4, 0) lies on the horizontal axis; thus implying that their convex combination is a downward diagonal line convexity the two points, as depicted in the following figure.



For instance, for intermediate values of λ (such as $\lambda = \frac{1}{2}$) we have that

$$\max\left\{\frac{1}{2}0 + \frac{1}{2}4, \frac{1}{2}4 + \frac{1}{2}0\right\} = \max\{2, 2\} = 2,$$

which does not exceed $\min\{3, 3\} = 3$. Hence, the preference relation is *not* convex.

5. *LNS.* Consider the bundle $(x_1, x_2) = (1, 0)$. To establish LNS we must find a pair $(y_1, y_2) \in \mathbb{R}^2$ such that (y_1, y_2) it is arbitrarily close to (x_1, x_2), and $(y_1, y_2) \succ (x_1, x_2)$ strictly. In order to obtain $(y_1, y_2) \succ (x_1, x_2)$ we need that $(y_1, y_2) \succsim (x_1, x_2)$ and $(x_1, x_2) \not\succsim (y_1, y_2)$. By the preference relation in this example, the first condition implies $\max\{y_1, y_2\} \geqslant \min\{x_1, x_2\}$, whereas the second condition implies $\max\{x_1, x_2\} < \min\{y_1, y_2\}$. However, $\min\{x_1, x_2\} = 0$ and $\max\{x_1, x_2\} = 1$. This implies that the above two conditions can be rewrite as

$$\max\{y_1, y_2\} \geqslant 0 \text{ and } 1 < \min\{y_1, y_2\}$$

or, more compactly, as $\max\{y_1, y_2\} \geqslant \min\{y_1, y_2\} > 1$. As a consequence, *both* coordinates in bundle (y_1, y_2) must exceed 1 for this condition to be fulfilled, and points to the northeast of $(1, 1)$ cannot be found to be arbitrarily close to $(x_1, x_2) = (1, 0)$. Hence, this preference relation does *not* satisfy LNS.

3. **Strictly Convex Preferences.** Consider strictly convex preferences defined on the consumption set $X = \mathbb{R}^2_+$.

(a) Suppose Alex has a utility function $U(x) = (1 + x_1)(1 + x_2)$. Show that his preferences are convex. Are his preferences strictly convex?

- The logarithm of $U(x)$ is well defined over $\mathbb{R}^2_+$. Define

$$u(x) = \ln U(x) = \ln(1 + x_1) + \ln(1 + x_2).$$

Because each term on the right-hand side is concave it follows from the previous exercise that preferences are convex. Indeed, because $\ln(1 + x_j)$ is a strictly concave function, $u(x)$ is strictly concave and so preferences are strictly convex.

(b) Bev has a utility function $U(x) = x_1 x_2$. Are her preferences convex or strictly convex?

- For all $x > 0$ the logarithm of $U(x)$ is well defined and strictly concave. Then preferences are strictly convex. It follows that for all strictly positive bundles x^0 and x^1, with convex combination $x^\lambda \equiv (1 - \lambda)x^0 + \lambda x^1$ where $0 < \lambda < 1$, such that $U(x^1) \geq U(x^0)$, we must have

$$U(x^\lambda) > U(x^0).$$

Thus the utility function is quasi-concave and preferences are convex.

- Note that if bundles x^0 and x^1 contain only positive amounts of opposite goods, i.e., $x^0 = (0, a)$ and $x^1 = (0, b)$ where $a, b > 0$, then utility levels of consuming these bundles are all zero,

$$U(x^0) = U(x^1) = 0$$

as well as the utility level of their convex combination, $x^\lambda \equiv (1 - \lambda)x^0 + \lambda x^1$, i.e., $U(x^\lambda) = 0$. Hence, that preferences are not strictly convex.

4. **Quasi-Linear Preference.** Write the $(n + 1)-$dimensional consumption vector x as (y, z) where y is a scalar and z is an $n-$dimensional consumption vector. A utility function $U(x)$ is quasi-linear if it can be written as follows $U(x) = \alpha y + V(z)$. The consumption set $X = \mathbb{R}^{n+1}_+$.

(a) Show that if V is concave, U is quasi-concave.

- Because a linear function is concave it follows that if $V(z)$ is concave, the sum $U(x) = \alpha y + V(z)$ is concave and hence quasi-concave.

(b) Show that if U is quasi-concave, V is concave.

- *Hint*: Suppose that for some x^0, x^1, x^λ, concavity fails; that is, $V(x^\lambda) < (1 - \lambda)V(x^0) + \lambda V(x^1)$. Choose y^0, y^1 such that $U(x^0) = U(x^1)$ and show that $U(x^\lambda) < U(x^0)$.
- For any vectors z^0, z^1, choose y^0, y^1 so that the utility level of (y^0, z^0) coincides with that of (y^1, z^1), that is

$$U(y^0, z^0) = y^0 + V(z^0) = y^1 + V(z^1) = U(y^1, z^1).$$

If U is quasi-concave, then $U(y^\lambda, z^\lambda) \geq U(y^0, z^0)$ and $U(y^\lambda, z^\lambda) \geq U(y^1, z^1)$, where $y^\lambda \equiv (1 - \lambda)y^0 + \lambda y^1$ and $z^\lambda \equiv (1 - \lambda)z^0 + \lambda z^1$.

- Multiplying the first inequality by $(1-\lambda)$ and the second by λ, we obtain

$$\begin{aligned} U(y^\lambda, z^\lambda) &= y^\lambda + V(z^\lambda) \geq (1-\lambda)(y^0 + V(z^0)) + \lambda(y^1 + V(z^1)) \\ &= y^\lambda + (1-\lambda)V(z^0) + \lambda V(z^1). \end{aligned}$$

Subtracting y^λ from both sides, it follows that V is concave.

5. **Rubinstein.** Problem set 2, Exercise 3.

- Let us construct inductively the sequence $\{(x_n, z_n, m_n)\}$ as follows: Let us start by defining initial points $x_0 = x$ and $z_0 = z$, and their midpoint $m_0 = \frac{1}{2}x_0 + \frac{1}{2}z_0$. Several possibilities must be considered when comparing the midpoint m_n and any bundle $y \in X$.
  - If $m_n \sim y$, then we found the point that we looked for.
  - If $m_n \succ y$, let $x_{n+1} = m_n$, $z_{n+1} = z_n$, and their midpoint $m_{n+1} = \frac{1}{2}x_{n+1} + \frac{1}{2}z_{n+1}$. In this case, $x_n \succ y \succ z_n$ for all n. In addition, both sequences (x_n) and (z_n) converge to some midpoint m^* which lies between x and z. Therefore, since the preference relation is continuous, then $m^* \succsim y$ and $y \succsim m^*$, which thus implies $m^* \sim y$, as required.
  - If $y \succ m_n$, let $x_{n+1} = x_n$, $z_{n+1} = m_n$, and their midpoint $m_{n+1} = \frac{1}{2}x_{n+1} + \frac{1}{2}z_{n+1}$. (A similar argument as above applies, producing $m^* \sim y$, as required.)

6. **Rubinstein.** Problem set 2, Exercise 4.

- Let $x \succ^* y$, which entails that $\max\{x_1, x_2\} > \max\{y_1, y_2\}$. Therefore, we can find a scalar $\epsilon > 0$ such that

$$\max\{x_1, x_2\} > (1+\epsilon)\max\{y_1, y_2\}.$$

Hence, when n is large enough, we have

$$[\max\{x_1, x_2\}]^n > 2\,[\max\{y_1, y_2\}]^n.$$

Finally,

$$x_1^n + x_2^n \geq [\max\{x_1, x_2\}]^n > 2\,[\max\{y_1, y_2\}]^n \geq y_1^n + y_2^n$$

which implies that $x \succ^n y$ for n large enough.

7. **Rubinstein.** Problem set 2, Exercise 5.

- See scanned pages at the end of this answer key.

8. **Rubinstein.** Problem set 2, Exercise 6.

- See scanned pages at the end of this answer key.

Problem 5.

Let X be a finite set and let $(\succsim, \succ\succ)$ be a pair where $\succsim$ is a preference relation and $\succ\succ$ is a transitive sub-relation of $\succ$ (by sub-relation, we mean $x \succ\succ y$ implies $x \succ y$). We can think about the pair as representing the responses to the questionnaire A where $A(x,y)$ is the question:

How do you compare x and y? Tick one of the following five options:

☐ **I very much prefer x over y** $(x \succ\succ y)$

☐ **I prefer x over y** $(x \succ y)$

☐ **I am indifferent** (I)

☐ **I prefer y over x** $(y \succ x)$

☐ **I very much prefer y over x** $(y \succ\succ x)$

Assume that the pair satisfies extended transitivity: If $x \succ\succ y$ and $y \succsim z$, or if $x \succsim y$ and $y \succ\succ z$ then $x \succ\succ z$. We say that a pair $(\succsim, \succ\succ)$ is represented by a function u if

$u(x) = u(y)$ **iff** $x \sim y$,

$u(x) - u(y) > 0$ **iff** $x \succ y$, **and**

$u(x) - u(y) > 1$ **iff** $x \succ\succ y$.

Show that every extended preference $(\succsim, \succ\succ)$ can be represented by a function u.

Denote $A \succ B$ if $a \succ b$ for all $a \in A$ and $b \in B$. Let $X_1, X_2, \ldots, X_K$ be the $\succsim$ indifference sets such that $X_K \succ X_{K-1} \succ \ldots \succ X_1$. Define first $u(X_1) = 0$.

Let us define $u(X_k)$ for $k > 1$.

(1) if $X_k \succ\succ X_{k-1}$, then $u(X_k) = u(X_{k-1}) + 2$

(2) if X_k is not $\succ\succ$ even of X_1, then $u(X_k) \in (u(X_{k-1}), 1)$

(3) otherwise, there exists a maximal $m(k)$ such that $X_k \succ\succ X_{m(k)}$. Define $u(X_k)$ such that $u(X_k) > u(X_{k-1})$ and $1 + u(X_{m(k)+1}) > u(X_k) > u(X_{m(k)}) + 1$.

Clearly, $x \sim y$ iff $u(x) = u(y)$

Also, if $x \succ y$ then $u(x) > u(y)$, since we picked $u(X_k)$ as an increasing sequence.

Finally, if $x \succ\succ y$, $x \in X_k$ and $y \in X_m$ then $m(k) \geq m$ and $u(x) > u(X_{m(k)}) + 1 \geq u(y) + 1$.

Problem 6.

The following is a typical example of a utility representation theorem: Let $X = \Re_+^2$. Assume that a preference relation $\succsim$ satisfies the following three properties:

ADD: $(a_1,a_2) \succsim (b_1,b_2)$ **implies that** $(a_1+t,a_2+s) \succsim (b_1+t,b_2+s)\ \forall s,t$.

SMON: **If** $a_1 \geq b_1$ **and** $a_2 \geq b_2$, **then** $(a_1,a_2) \succsim (b_1,b_2)$. **In addition, if either** $a_1 > b_1$ **or** $a_2 > b_2$ **then** $(a_1,a_2) \succ (b_1,b_2)$.

CON: **Continuity.**

a. Show that if $\succsim$ has a linear representation (that is, $\succsim$ are represented by a utility function $u(x_1,x_2) = \alpha x_1 + \beta x_2$ with $\alpha,\beta > 0$), then $\succsim$ satisfies ADD, SMON, CON.

ADD: Let $s,t \in \Re$ and $x,y \in X$ be such that $x \succsim y$. Note that
$(x_1,x_2) \succsim (y_1,y_2) \Leftrightarrow \alpha x_1 + \beta x_2 \geq \alpha y_1 + \beta y_2 \Leftrightarrow \alpha(x_1+t) + \beta(x_2+s) \geq \alpha(y_1+t) + \beta(y_2+s) \Leftrightarrow$
$u(x_1+t,x_2+s) \geq u(y_1+t,y_2+s) \Leftrightarrow (x_1+t,x_2+s) \succsim (y_1+t,y_2+s)$.

SMON: Let $x,y \in X$ be such that $x_1 \geq y_1$ and $x_2 \geq y_2$ with at least one strict inequality. Since $\alpha,\beta > 0$, then $\alpha x_1 + \beta x_2 > \alpha y_1 + \beta y_2$, which implies that $(x_1,x_2) \succ (y_1,y_2)$.

CON: $u(x_1,x_2)$ is continuous, and thus $\succsim$ is continuous.

b. Show that for any pair of the three properties there is a preference relation that does not satisfy the third property.

Satisfies only ADD, SMON: Lexicographic preferences satisfy *ADD* and *SMON*, but are not continuous (see the lecture notes).

Satisfies only ADD, CON: The preferences represented by $u(x_1,x_2) = x_1 - x_2$ satisfy *ADD* and *CON*, but not *SMON* since $(1,1) \succ (1,2)$.

Satisfies only MON, CON: Preferences represented by $u(x_1,x_2) = x_1^2 + x_2^2$ satisfy *SMON* and *CON*, but not *ADD* since $(3,0) \succ (2,1)$ and $(3,3) \prec (2,4)$.

c. Show that if $\succsim$ satisfies the three properties, then it has a linear representation.

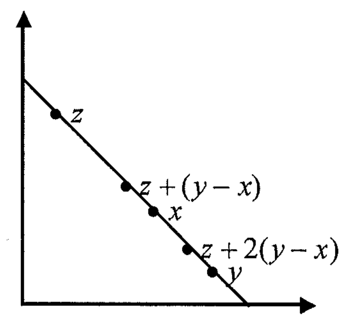
Assume first that x and y are two different points such that $x \sim y$. Then:

(i) $(x+y)/2 \sim y$. Otherwise, $(x+y)/2 \succ y$ would imply that
$x = \frac{x+y}{2} + \frac{x-y}{2} \succ y + \frac{x-y}{2} = \frac{x+y}{2} \succ y$ by ADD, a contradiction.

(ii) $z = (1-\alpha)x + \alpha y \sim x$ for $\alpha \in [0,1]$. Define $\{(x^n,y^n)\}$ inductively as follows: let $x^0 = x$, $y^0 = y$. Let $m^0 = (x^0+y^0)/2$.

Assume z belongs to $[x^n,y^n]$ and its length is $1/2^n$ the length of $[x,y]$. The point z belongs to at least one of the intervals $[x^n,m^n]$ and $[m^n,y^n]$. Define $[x^{n+1},y^{n+1}]$ to be one of those intervals which contains z. Now, all $x^n \sim x$ for all n. The sequence $x^n \to z$, therefore by continuity $z \sim x$.

(iii) Let z be on the line which connects x and y, $z \sim x$. Without loss of generality, assume that z is closer to x. There is n such that $w = z + n(y-x)$ is between x and y. By ADD if $a - x = b - y$ (that is a-b=x-y) then $a \sim b$. Thus by transitivity $z \sim w \sim x$.



By SMON there is an $\varepsilon > 0$ such that $a = (x_1 + \varepsilon, x_2) \succ x \succ (x_1, x_2 - \varepsilon) = b$. By question 3, there exists y (different than x) on the interval which connects a and b such that $x \sim y$. Thus, every point is on a difference line which is a line. The indifference lines must be parallel since otherwise we will get a contradiction to *ADD*.

d. Characterize the preference relations which satisfy ADD, SMON and an additional property MUL:

$(a_1, a_2) \succsim (b_1, b_2)$ *implies that* $(\lambda a_1, \lambda a_2) \succsim (\lambda b_1, \lambda b_2)$ *for any* $\lambda \geq 0$.

Define $s = sup\{x | (0,1) \succ (x,0)\}$ (by SMON the set is not empty).

Case (1): $s = \infty$ or $s = 0$: the preferences must be lexicographic with priority for the second or first components, respectively.

Assume $s = \infty$.

If $a_2 > b_2$ then $(a_1, a_2) \succ (b_1, b_2)$ iff $(a_1, a_2 - b_2) \succ (b_1, 0)$ (by ADD) iff $(a_1/(a_2 - b_2), 1) \succ (b_1/(a_2 - b_2), 0)$ (by MUL), which is always true (by $s = \infty$).

If $a_2 = b_2$ then $(a_1, a_2) \succ (b_1, b_2)$ iff $a_1 > b_1$ (by SMON).

Thus, we have a lexicographic relation with priority for the second component.

If $s = 0$ then it follows that $s = sup\{y | (1,0) \succ (0,y)\} = \infty$ and the preferences must be lexicographic with priority for the first component.

Case (2): $\infty > s > 0$

Let (a_1, a_2) and (b_1, b_2) be two vectors with $a_1 \leq b_1$. (a_1, a_2) relates to (b_1, b_2) as $(0, a_2 - b_2)$ relates to $(b_1 - a_1, 0)$ (by ADD) and thus as $((b_1 - a_1)/(a_2 - b_2), 0)$ relates to $(0,1)$. This relation is determined by the comparison of $(b_1 - a_1)/(a_2 - b_2)$ to s, which is equivalent to the comparison of $a_1 + sa_2$ and $b_1 + sb_2$.

Therefore, if $(0,1) \sim (s,0)$ then $x_1 + sx_2$ represents the preferences. If $(0,1) \succ (s,0)$ or

$(0,1) \prec (s,0)$ then the preferences are lexicographic with the first priority to $x_1 + sx_2$ and the second to x_2 or x_1 accordingly.