Core and Equilibria

- While we know that WEAs are part of the Core...
- In this section we seek to show that, as the economy becomes larger, the Core shrinks until it exactly coincides with the set of WEAs.
  - Section 5.5 in JR.
- Consider an economy with I consumers, each with $\left(u^i, \mathbf{e}^i\right)$.
  - Now consider its replica: we double the number of consumers to $2I$, each of them still with $\left(u^i, \mathbf{e}^i\right)$.
  - There are now two consumers of each type, i.e., "twins," having identical preferences and endowments.

Core and Equilibria

- We can now define an **r-fold replica economy** $\mathcal{E}_r$**:**
  - $\mathcal{E}_r$ has r consumers of each type, for a total of rI consumers.
  - In addition, for any type $i \in I$, all r consumers of that type share the common utility function u^i and have identical endowments $\mathbf{e}^i >> 0$.
- When comparing two replica economies, the largest will be that having more of every type of consumer.

Core and Equilibria

- Let us now examine the core of the replica economy $\mathcal{E}_r$:
  - From our assumptions on consumer preferences, we know that the WEA will exist, and that it will be in the Core.
  - Then, the core of the replica economy $\mathcal{E}_r$ will exist.

Core and Equilibria

- Notation:
  - Allocation $\mathbf{x}^{iq}$ indicates the vector of goods for the qth consumer of type i (you can think about consumer i existing in the original economy, and now having $r > q$ twins in the r-fold replica economy).
  - Given this notation, we can rewrite feasibility in this setting as follows:

$$\sum_{i=1}^{I} \sum_{q=1}^{r} \mathbf{x}^{iq} = r \sum_{i=1}^{I} \mathbf{e}^{i}$$

    since each of the r consumers of type i has a endowment vector $\mathbf{e}^{i}$.

- Not only similar types start with the same endowment vector $\mathbf{e}^{i}$, but they also end up with the same allocation at the Core (next slide).

Core and Equilibria

- **Equal treatment at the Core:**
  - If $\mathbf{x}$ is an allocation in the Core of $\mathcal{E}_r$, then every consumer of type i must have the same bundle, i.e.,

    $$\mathbf{x}^{iq} = \mathbf{x}^{iq'}$$

    for every two "twins" q and q' of type i, $q \neq q' \in \{1, 2, ..., r\}$, and for every type $i \in I$.
  - **Proof:**
  - We will prove the above result for a two-fold replica economy, $\mathcal{E}_2$. You can easily generalize it to r-fold replicas.
  - Suppose that allocation

    $$\mathbf{x} \equiv \left\{ \mathbf{x}^{11}, \mathbf{x}^{12}, \mathbf{x}^{21}, \mathbf{x}^{22} \right\}$$

    is an allocation at the core of $\mathcal{E}_2$ (as required in the premise of the above claim).

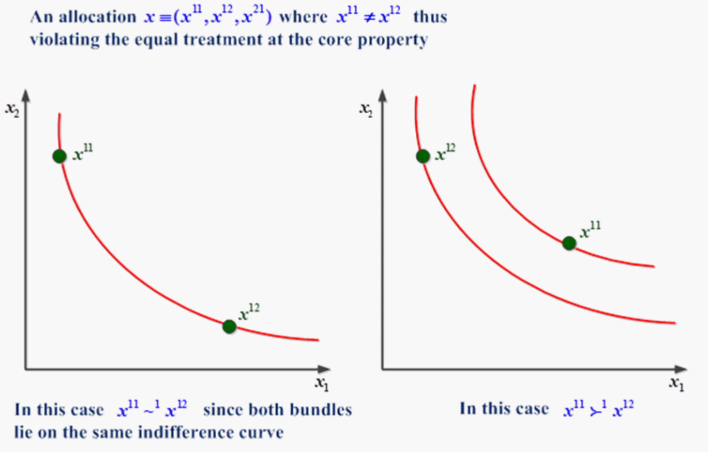
Core and Equilibria

- **Equal treatment at the Core:**
  - **Proof (cont'd):**
  - Since $\mathbf{x}$ is in the core, then it must be feasible

$$\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22} = 2\mathbf{e}^{1} + 2\mathbf{e}^{2}$$

    because the two type-1 consumers have identical endowments, and so do the two type-2 consumers.
  - By contradiction: assume now that $\mathbf{x}$, despite being at the core, does not assign the same consumption vectors to the two twins of type-1, i.e., $\mathbf{x}^{11} \neq \mathbf{x}^{12}$.
  - WLOG $\mathbf{x}^{11} \succsim^{1} \mathbf{x}^{12}$ which is true for both type-1 twins, since they have the same preferences.
    - Figures depicting $\mathbf{x}^{11} \succsim^{1} \mathbf{x}^{12}$ and $\mathbf{x}^{11} \succ^{1} \mathbf{x}^{12}$.

Core and Equilibria



Core and Equilibria

- **Equal treatment at the Core:**
  - **Proof (cont'd):**
  - Before going any further: What are we looking for?
  - If we are operating by contradiction, we need that...
    - When the premise of the claim is satisfied ($\mathbf{x}$ is at the core) but the conclusion is violated (*unequal* treatment at the core, $\mathbf{x}^{11} \neq \mathbf{x}^{12}$),
    - We end up with the original premise being contradicted (i.e., $\mathbf{x}$ is *not* at the core because we can find a blocking coalition).

- **Equal treatment at the Core:**
  - **Proof (cont'd):**
  - In designing a potential blocking coalition, consider that for type-2 consumers we have $\mathbf{x}^{21} \succsim^2 \mathbf{x}^{22}$.
    - (This is done WLOG, since the same result would apply if we revert this preference relation, making consumer 1 of type 2 the worst off.)
  - Hence, consumer 2 of type 1 is the worst off type 1 consumer, i.e., $\mathbf{x}^{11} \succsim^1 \mathbf{x}^{12}$, and consumer 2 of type 2 is the worst off type 2 consumer.
  - Let's take these two "poorly treated" consumers, and check if they can form a blocking coalition to oppose $\mathbf{x}$.

Core and Equilibria

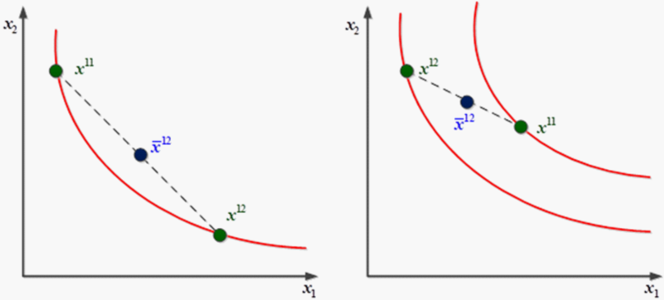
- **Equal treatment at the Core:**
  - **Proof (cont'd):**
  - Define the average bundles

$$\bar{\mathbf{x}}^{12} = \frac{\mathbf{x}^{11} + \mathbf{x}^{12}}{2} \quad \text{and} \quad \bar{\mathbf{x}}^{22} = \frac{\mathbf{x}^{21} + \mathbf{x}^{22}}{2}$$

    where the first (second) bundle is the average of the bundles going to the type-1 (type-2, respectively) consumers.
  - See figure in next slide for the location of these bundles.

Core and Equilibria

Finding a blocking coalition to $x \equiv (x^{11}, x^{12}, x^{21})$ where $x^{11} \neq x^{12}$



In this case $x^{11} \sim^{1} x^{12}$ but we can find another bundle, $\bar{x}^{12}$, which satisfies $\bar{x}^{12} \succ^{1} x^{12}$

In this case $x^{11} \succ^{1} x^{12}$, but we can still find another bundle, $\bar{x}^{12}$, which satisfies $\bar{x}^{12} \succ^{1} x^{12}$

Core and Equilibria

- **Equal treatment at the Core:**
  - **Proof (cont'd):**
  - Because of preferences being strictly convex, the worst off type-1 consumer prefers

$$\bar{\mathbf{x}}^{12} \succ^{1} \mathbf{x}^{12},$$

    since $\bar{\mathbf{x}}^{12}$ is a linear combination between $\mathbf{x}^{11}$ and his original bundle $\mathbf{x}^{12}$. (See previous figures.)
  - A similar argument applies to the worst off type-2 consumer, $\bar{\mathbf{x}}^{22} \succ^{2} \mathbf{x}^{22}$.
  - We have now found a pair of bundles $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$, which would both consumers 12 and 22 better off than at the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$.
  - The question that still remains is: Can they achieve this pair of bundles, i.e., is $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$ feasible?

Core and Equilibria

- **Equal treatment at the Core:**
  - **Proof (cont'd):**
  - Finally checking for the **feasibility of the pair of bundles** $\left(\overline{\mathbf{x}}^{12}, \overline{\mathbf{x}}^{22}\right)$.
  - We can rewrite the amount of goods they need to achieve $\left(\overline{\mathbf{x}}^{12}, \overline{\mathbf{x}}^{22}\right)$ as follows:

$$\begin{aligned} \overline{\mathbf{x}}^{12} + \overline{\mathbf{x}}^{22} &= \frac{\mathbf{x}^{11} + \mathbf{x}^{12}}{2} + \frac{\mathbf{x}^{21} + \mathbf{x}^{22}}{2} \\ &= \frac{1}{2}\left(\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22}\right) \\ &= \frac{1}{2}\left(2\mathbf{e}^{1} + 2\mathbf{e}^{2}\right) \\ &= \mathbf{e}^{1} + \mathbf{e}^{2} \end{aligned}$$

Core and Equilibria

- **Equal treatment at the Core:**
  - **Proof (cont'd):**
  - Hence, the pair of bundles $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$ is feasible.
  - Since this pair of bundles makes the consumers 12 and 22 better off than at the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$, and $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$ is feasible, these consumers will get together to block $(\mathbf{x}^{12}, \mathbf{x}^{22})$.
  - As a consequence, the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$ cannot be at the Core, since we found a blocking coalition.
  - Then, if an allocation is at the Core of the replica economy, it must give consumers of the same type the same bundle.

Core and Equilibria

- After proving the "equal treatment at the core" property...
- We are ready to continue with our main goal of this section:
  - As the economy becomes larger (r increases), the Core shrinks, and if r is sufficiently large the Core converges to the set of WEAs.

Core and Equilibria

- *Remark*:
  - The "equal treatment at the core" property helps us describe core allocations in a r-fold replica economy $\mathcal{E}_r$ by reference to a similar allocation in the original (unreplicated) economy $\mathcal{E}_1$
  - In particular, if $\mathbf{x}$ is in the core of a r-fold replica economy $\mathcal{E}_r$, then by the equal treatment property, allocation $\mathbf{x}$ must be of the form

$$\mathbf{x} = \left(\underbrace{\mathbf{x}^1, ..., \mathbf{x}^1}_{r \text{ times}}, \underbrace{\mathbf{x}^2, ..., \mathbf{x}^2}_{r \text{ times}}, ..., \underbrace{\mathbf{x}^I, ..., \mathbf{x}^I}_{r \text{ times}} \right)$$

because all consumers of the same type must receive the same bundle.

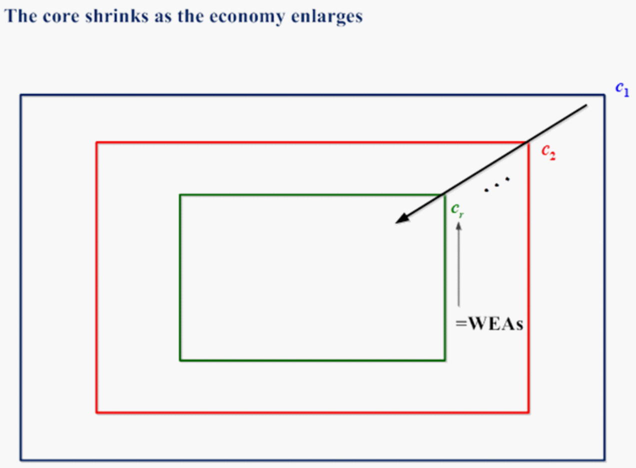
Core and Equilibria

- Therefore, core allocations in $\mathcal{E}_r$ are just r-fold copies of allocations in $\mathcal{E}_1$, $\mathbf{x} = \left(\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^I\right)$.
    - *Notation*: We define the core in $\mathcal{E}_r$ as C_r.
- We can now show that, as r increases, the core shrinks.

Core and Equilibria

- **The core shrinks as the economy enlarges:**
  - The sequence of core sets C_1, C_2,... is decreasing.
  - That is, the core of the original (unreplicated) economy, C_1, is a superset of that in the 2-fold replica economy, C_2.
  - In addition, the core in the 2-fold replica economy, C_2, is a superset of the 3-fold replica economy, C_3; etc.
  - More compactly, $C_1 \supseteq C_2 \supseteq C_3 \supseteq ... \supseteq C_r \supseteq ...$
- Silly figure, and then proof.

Core and Equilibria

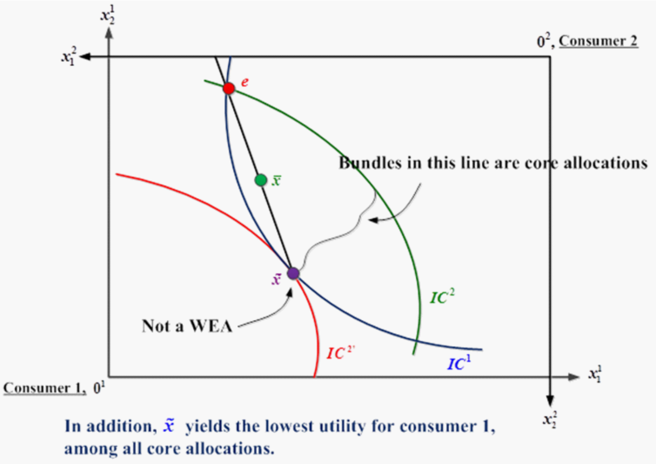


- **The core shrinks as the economy enlarges:**
  - *Proof:*
  - It suffices to show that, for any $r > 1$, $C_{r-1} \supseteq C_r$.
  - First, suppose that allocation $\mathbf{x} = \left(\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^I\right) \in C_r$. Intuitively, we cannot find any blocking coalition to $\mathbf{x}$ in the r-fold replica economy $\mathcal{E}_r$.
  - We now need to show that $\mathbf{x}$ cannot be blocked by any coalition in the $(r-1)$-fold replica economy $\mathcal{E}_{r-1}$ either.
  - But if could find a blocking coalition to $\mathbf{x}$ in $\mathcal{E}_{r-1}$ then we could also find a blocking coalition in $\mathcal{E}_r$:
    - Indeed, all members in $\mathcal{E}_{r-1}$ are also present in the larger economy $\mathcal{E}_r$ and their endowments haven't changed.

Core and Equilibria

- **The core shrinks as the economy enlarges:**
  - *Graphical representation*
  - See next figure.
  - In the unreplicated economy $\mathcal{E}_1$ the set of core allocations is the line between $\widetilde{\mathbf{x}}$ and $\mathbf{e}$
  - Some point in the line connecting $\widetilde{\mathbf{x}}$ and $\mathbf{e}$ are WEAs and some aren't.
    - For instance, $\widetilde{\mathbf{x}}$ is not a WEA: the price line through $\widetilde{\mathbf{x}}$ and $\mathbf{e}$ is not tangent to $\widetilde{\mathbf{x}}$ to the consumer's indifference curve at $\widetilde{\mathbf{x}}$.
    - In addition, note that allocation $\widetilde{\mathbf{x}}$, despite being at the core, yields the same utility level as endowment $\mathbf{e}$ for consumer 1. That is, is the "worst" admissible allocation for consumer 1 among all core allocations.

Core and Equilibria



Core and Equilibria

- **The core shrinks as the economy enlarges:**
  - *Question:* Does allocation $\widetilde{\mathbf{x}}$ remain at the core of the two-fold replica economy $\mathcal{E}_2$?
  - No!
    - In particular, any point on the line connecting $\widetilde{\mathbf{x}}$ and $\mathbf{e}$ is strictly preferred by both types of consumer 1 (he now has a twin!).
  - Let's next try to build a blocking coalition against $\widetilde{\mathbf{x}}$:
    - We will need to guarantee:
    - *Acceptance* by all coalition members, and
    - *Feasibility* of the proposed allocation.

Core and Equilibria

- **The core shrinks as the economy enlarges:**
  - Building a blocking coalition against $\widetilde{\mathbf{x}}$:
    - Consider the midpoint allocation $\overline{\mathbf{x}}$ and the coalition $S = \{11, 12, 21\}$.
    - *Acceptance:* If the midpoint allocation $\overline{\mathbf{x}}$ is offered to 11 and 12, and the content in $\widetilde{\mathbf{x}}$ is offered to 21, will they accept? Yes:

$$\begin{aligned}
\overline{\mathbf{x}}^{11} &\equiv \frac{1}{2}\left(\mathbf{e}^{1} + \widetilde{\mathbf{x}}^{11}\right) \succ^{1} \widetilde{\mathbf{x}}^{11}, \\
\overline{\mathbf{x}}^{12} &\equiv \frac{1}{2}\left(\mathbf{e}^{1} + \widetilde{\mathbf{x}}^{12}\right) \succ^{1} \widetilde{\mathbf{x}}^{12}, \\
\widetilde{\mathbf{x}}^{21} &\sim^{2} \widetilde{\mathbf{x}}^{21}
\end{aligned}$$

Core and Equilibria

- **The core shrinks as the economy enlarges:**
  - Building a blocking coalition against $\widetilde{\mathbf{x}}$ (cont'd):
  - *Feasibility:* Let us now check that the suggested allocation $\left\{\overline{\mathbf{x}}^{11}, \overline{\mathbf{x}}^{12}, \widetilde{\mathbf{x}}^{21}\right\}$ is feasible for coalition S.
    - Since $\overline{\mathbf{x}}^{11} = \overline{\mathbf{x}}^{12}$, then the sum of the suggested allocation yields

$$\begin{aligned} \overline{\mathbf{x}}^{11} + \overline{\mathbf{x}}^{12} + \widetilde{\mathbf{x}}^{21} &= 2\frac{1}{2}\left(\mathbf{e}^{1} + \widetilde{\mathbf{x}}^{11}\right) + \widetilde{\mathbf{x}}^{21} \\ &= \mathbf{e}^{1} + \widetilde{\mathbf{x}}^{11} + \widetilde{\mathbf{x}}^{21} \end{aligned}$$

    - Recall now that $\widetilde{\mathbf{x}}$ was part of the unreplicated economy $\mathcal{E}_1$. It then must be feasible, i.e., $\widetilde{\mathbf{x}}^{1} + \widetilde{\mathbf{x}}^{2} = \mathbf{e}^{1} + \mathbf{e}^{2}$. Hence, $\widetilde{\mathbf{x}}^{11} + \widetilde{\mathbf{x}}^{21} = \mathbf{e}^{1} + \mathbf{e}^{2}$.

Core and Equilibria

- **The core shrinks as the economy enlarges:**
  - Building a blocking coalition against $\widetilde{\mathbf{x}}$ (cont'd):
    - Combining the above two results, we obtain

$$\begin{aligned}
\overline{\mathbf{x}}^{11} + \overline{\mathbf{x}}^{12} + \widetilde{\mathbf{x}}^{21} &= \mathbf{e}^1 + \underbrace{\widetilde{\mathbf{x}}^{11} + \widetilde{\mathbf{x}}^{21}}_{\mathbf{e}^1 + \mathbf{e}^2} \\
&= \mathbf{e}^1 + \mathbf{e}^1 + \mathbf{e}^2 \\
&= 2\mathbf{e}^1 + \mathbf{e}^2
\end{aligned}$$

Thus confirming feasibility.

- **WEA in replicated economies.**
  - Consider a WEA in the unreplicated economy $\mathcal{E}_1$,

$$\left(\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^I\right)$$

  - Then, an allocation $\mathbf{x}$ is a WEA for the r-fold replica economy $\mathcal{E}_r$ if and only if it is of the form

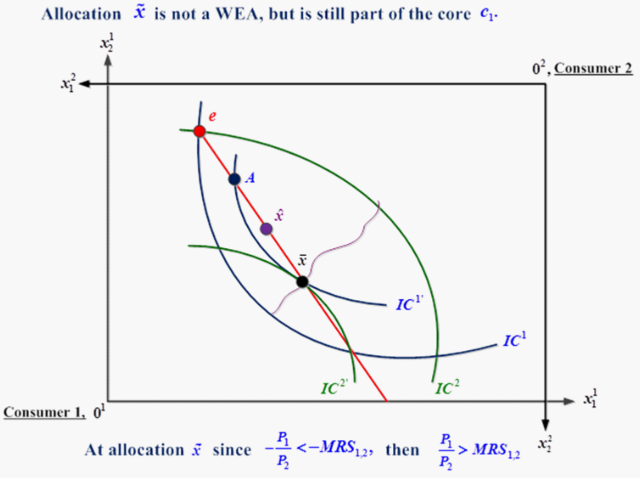
$$\mathbf{x} = \left(\underbrace{\mathbf{x}^1, ..., \mathbf{x}^1}_{r \text{ times}}, \underbrace{\mathbf{x}^2, ..., \mathbf{x}^2}_{r \text{ times}}, ..., \underbrace{\mathbf{x}^I, ..., \mathbf{x}^I}_{r \text{ times}}\right)$$

  - *Proof:* If $\mathbf{x}$ is a WEA for $\mathcal{E}_r$, then it also belongs to the core of $\mathcal{E}_r$. By the "equal treatment at the core" property, the result follows.

Core and Equilibria

- We are now ready to present the main result of this section.
- **A limit theorem on the Core:**
  - If allocation $\mathbf{x}$ is in the Core of the r-fold replica economy $\mathcal{E}_r$, for every $r \geq 1$, then $\mathbf{x}$ is a WEA for the unreplicated economy $\mathcal{E}_1$.
- Let's consider, by contradiction, that an allocation $\widetilde{\mathbf{x}}$ is not a WEA, but still belongs to the core of the r-fold replica economy C_r; see next figure.
- Then, $\widetilde{\mathbf{x}} \in C_1$ since $C_1 \supset C_r$.
  - In the next figure, this means that allocation $\widetilde{\mathbf{x}}$ must be within the lens and on the contract curve.

Core and Equilibria



Core and Equilibria

- Consider the line connecting $\widetilde{\mathbf{x}}$ and $\mathbf{e}$.
- Since $\widetilde{\mathbf{x}} \notin W(\mathbf{e})$, then either $\frac{p_1}{p_2} > MRS$ or $\frac{p_1}{p_2} < MRS$.
  - (The figure depicts the first case; the second is analogous.)
- By convexity of preferences, we can find a set of bundles, such as those between A and $\widetilde{\mathbf{x}}$ in the figure, that consumer 1 prefers to $\widetilde{\mathbf{x}}$.
- One example of such bundle is the linear combination

$$\widehat{\mathbf{x}} \equiv \frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\widetilde{\mathbf{x}}^1$$

for some $r > 1$, where $\frac{1}{r} + \frac{r-1}{r} = 1$.

Core and Equilibria

- The question we now pose is, can allocation $\widetilde{\mathbf{x}}$ be at the core of the r-fold replica economy $\mathcal{E}_r$ if it is not a WEA?
  - No, if we can find a blocking coalition.
  - Consider a coalition S with all r type 1 consumers and $r-1$ type 2 consumers.
  - *Acceptance*:
    - If we give every type 1 consumer the bundle $\widehat{\mathbf{x}}^1$, we know that $\widehat{\mathbf{x}}^1 \succ^1 \widetilde{\mathbf{x}}^1$
    - If we give every type 2 consumer in the coalition the bundle $\widehat{\mathbf{x}}^2 = \widetilde{\mathbf{x}}^2$, then $\widehat{\mathbf{x}}^2 \sim^2 \widetilde{\mathbf{x}}^2$.

Core and Equilibria

- *Feasibility:*
  - Summing over the consumers in coalition S, their aggregate allocation is

$$\begin{aligned} r\widehat{\mathbf{x}}^1 + (r-1)\widehat{\mathbf{x}}^2 &= r\left[\frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\widetilde{\mathbf{x}}^1\right] + (r-1)\widetilde{\mathbf{x}}^2 \\ &= \mathbf{e}^1 + (r-1)\left(\widetilde{\mathbf{x}}^1 + \widetilde{\mathbf{x}}^2\right) \end{aligned}$$

  - Since $\widetilde{\mathbf{x}} \equiv \left(\widetilde{\mathbf{x}}^1, \widetilde{\mathbf{x}}^2\right)$ is in the core of the unreplicated economy $\mathcal{E}_1$, then it must be feasible

$$\widetilde{\mathbf{x}}^1 + \widetilde{\mathbf{x}}^2 = \mathbf{e}^1 + \mathbf{e}^2$$

Core and Equilibria

- *Feasibility* (cont'd)*:*
  - Combining the above two results, we find that

$$\begin{aligned} r\widehat{\mathbf{x}}^1 + (r-1)\widehat{\mathbf{x}}^2 &= \mathbf{e}^1 + (r-1)\underbrace{\left(\mathbf{e}^1 + \mathbf{e}^2\right)}_{\widetilde{\mathbf{x}}^1+\widetilde{\mathbf{x}}^2} \\ &= r\mathbf{e}^1 + r\left(\mathbf{e}^1 + \mathbf{e}^2\right) - \left(\mathbf{e}^1 + \mathbf{e}^2\right) \\ &= r\mathbf{e}^1 + (r-1)\mathbf{e}^2 \end{aligned}$$

    Thus confirming feasibility.
- Hence, r type 1 consumers and $r-1$ type 2 consumers can get together in coalition S, and block allocation $\widetilde{\mathbf{x}}$.
- Therefore, $\widetilde{\mathbf{x}}$ cannot be in the Core of the r-fold replica economy $\mathcal{E}_r$.
- Then, if $\widetilde{\mathbf{x}} \in C_r$, then $\widetilde{\mathbf{x}}$ must be a WEA.