While we know that WEAs are part of the Core...

In this section we seek to show that, as the economy becomes larger, the Core shrinks until it exactly coincides with the set of WEAs.

- Section 5.5 in JR.

Consider an economy with $I$ consumers, each with $(u^i, e^i)$.

- Now consider its replica: we double the number of consumers to $2I$, each of them still with $(u^i, e^i)$.
- There are now two consumers of each type, i.e., "twins," having identical preferences and endowments.
We can now define an **r-fold replica economy** $\mathcal{E}_r$:

- $\mathcal{E}_r$ has $r$ consumers of each type, for a total of $rl$ consumers.
- In addition, for any type $i \in I$, all $r$ consumers of that type share the common utility function $u^i$ and have identical endowments $e^i >> 0$.

When comparing two replica economies, the largest will be that having more of every type of consumer.
Let us now examine the core of the replica economy $\mathcal{E}_r$:

- From our assumptions on consumer preferences, we know that the WEA will exist, and that it will be in the Core.
- Then, the core of the replica economy $\mathcal{E}_r$ will exist.
Core and Equilibria

- **Notation:**
  - Allocation $x^{iq}$ indicates the vector of goods for the $q$th consumer of type $i$ (you can think about consumer $i$ existing in the original economy, and now having $r > q$ twins in the $r$-fold replica economy).
  - Given this notation, we can rewrite feasibility in this setting as follows:

  $\sum_{i=1}^{l} \sum_{q=1}^{r} x^{iq} = r \sum_{i=1}^{l} e^i$

  since each of the $r$ consumers of type $i$ has an endowment vector $e^i$.

- Not only similar types start with the same endowment vector $e^i$, but they also end up with the same allocation at the Core (next slide).
Equal treatment at the Core:

- If $x$ is an allocation in the Core of $\mathcal{E}_r$, then every consumer of type $i$ must have the same bundle, i.e.,

$$x^{iq} = x^{iq'}$$

for every two "twins" $q$ and $q'$ of type $i$, $q \neq q' \in \{1, 2, ..., r\}$, and for every type $i \in \mathcal{I}$.

Proof:

- We will prove the above result for a two-fold replica economy, $\mathcal{E}_2$. You can easily generalize it to $r$-fold replicas.

- Suppose that allocation

$$x \equiv \{x^{11}, x^{12}, x^{21}, x^{22}\}$$

is an allocation at the core of $\mathcal{E}_2$ (as required in the premise of the above claim).
Equal treatment at the Core:

- **Proof (cont’d):**
  - Since \( x \) is in the core, then it must be feasible
    \[
    x^{11} + x^{12} + x^{21} + x^{22} = 2e^1 + 2e^2
    \]
    because the two type-1 consumers have identical endowments, and so do the two type-2 consumers.
  - By contradiction: assume now that \( x \), despite being at the core, does not assign the same consumption vectors to the two twins of type-1, i.e., \( x^{11} \neq x^{12} \).
  - WLOG \( x^{11} \succsim^1 x^{12} \) which is true for both type-1 twins, since they have the same preferences.
    - Figures depicting \( x^{11} \succsim^1 x^{12} \) and \( x^{11} \succeq^1 x^{12} \).
Core and Equilibria

An allocation \( x = (x^{11}, x^{12}, x^{21}) \) where \( x^{11} \neq x^{12} \) thus violating the equal treatment at the core property.

In this case \( x^{11} \sim^1 x^{12} \) since both bundles lie on the same indifference curve.

In this case \( x^{11} \succ^1 x^{12} \).
Equal treatment at the Core:

Proof (cont’d):

Before going any further: What are we looking for?
If we are operating by contradiction, we need that...

- When the premise of the claim is satisfied ($x$ is at the core) but the conclusion is violated (*unequal* treatment at the core, $x^{11} \neq x^{12}$),
- We end up with the original premise being contradicted (i.e., $x$ is *not* at the core because we can find a blocking coalition).
Equal treatment at the Core:

Proof (cont’d):
In designing a potential blocking coalition, consider that for type-2 consumers we have $x^{21} \succeq^2 x^{22}$.

(This is done WLOG, since the same result would apply if we revert this preference relation, making consumer 1 of type 2 the worst off.)

Hence, consumer 2 of type 1 is the worst off type 1 consumer, i.e., $x^{11} \succeq^1 x^{12}$, and consumer 2 of type 2 is the worst off type 2 consumer.

Let’s take these two "poorly treated" consumers, and check if they can form a blocking coalition to oppose $x$. 
Core and Equilibria

- **Equal treatment at the Core:**
  - **Proof (cont’d):**
  - Define the average bundles

\[
\bar{x}^{12} = \frac{x^{11} + x^{12}}{2} \quad \text{and} \quad \bar{x}^{22} = \frac{x^{21} + x^{22}}{2}
\]

where the first (second) bundle is the average of the bundles going to the type-1 (type-2, respectively) consumers.
- See figure in next slide for the location of these bundles.
Finding a blocking coalition to $x \equiv (x^{11}, x^{12}, x^{21})$ where $x^{11} \neq x^{12}$

In this case $x^{11} \sim x^{12}$ but we can find another bundle, $\bar{x}^{12}$, which satisfies $\bar{x}^{12} \succ x^{12}$

In this case $x^{11} \succ x^{12}$, but we can still find another bundle, $\bar{x}^{12}$, which satisfies $\bar{x}^{12} \succ x^{12}$
Equal treatment at the Core:

Proof (cont’d):
Because of preferences being strictly convex, the worst off type-1 consumer prefers
\[ \overline{x}^{12} \succ^{1} x^{12}, \]

since \( \overline{x}^{12} \) is a linear combination between \( x^{11} \) and his original bundle \( x^{12} \). (See previous figures.)

A similar argument applies to the worst off type-2 consumer, \( \overline{x}^{22} \succ^{2} x^{22} \).

We have now found a pair of bundles \( (\overline{x}^{12}, \overline{x}^{22}) \), which would both consumers 12 and 22 better off than at the original allocation \( (x^{12}, x^{22}) \).

The question that still remains is: Can they achieve this pair of bundles, i.e., is \( (\overline{x}^{12}, \overline{x}^{22}) \) feasible?
Equal treatment at the Core:

Proof (cont’d):

Finally checking for the feasibility of the pair of bundles $(\bar{x}^{12}, \bar{x}^{22})$.

We can rewrite the amount of goods they need to achieve $(\bar{x}^{12}, \bar{x}^{22})$ as follows:

$$\bar{x}^{12} + \bar{x}^{22} = \frac{x^{11} + x^{12}}{2} + \frac{x^{21} + x^{22}}{2}$$

$$= \frac{1}{2} \left( x^{11} + x^{12} + x^{21} + x^{22} \right)$$

$$= \frac{1}{2} \left( 2e^1 + 2e^2 \right)$$

$$= e^1 + e^2$$
Equal treatment at the Core:

Proof (cont’d):

Hence, the pair of bundles \((\bar{x}^{12}, \bar{x}^{22})\) is feasible.

Since this pair of bundles makes the consumers 12 and 22 better off than at the original allocation \((x^{12}, x^{22})\), and \((\bar{x}^{12}, \bar{x}^{22})\) is feasible, these consumers will get together to block \((x^{12}, x^{22})\).

As a consequence, the original allocation \((x^{12}, x^{22})\) cannot be at the Core, since we found a blocking coalition.

Then, if an allocation is at the Core of the replica economy, it must give consumers of the same type the same bundle.
After proving the "equal treatment at the core" property...

We are ready to continue with our main goal of this section:

- As the economy becomes larger ($r$ increases), the Core shrinks, and if $r$ is sufficiently large the Core converges to the set of WEAs.
Remark:

- The "equal treatment at the core" property helps us describe core allocations in a $r$-fold replica economy $\mathcal{E}_r$ by reference to a similar allocation in the original (unreplicated) economy $\mathcal{E}_1$.
- In particular, if $x$ is in the core of a $r$-fold replica economy $\mathcal{E}_r$, then by the equal treatment property, allocation $x$ must be of the form

\[
x = \left( \underbrace{x^1, \ldots, x^1}_r, \underbrace{x^2, \ldots, x^2}_r, \ldots, \underbrace{x', \ldots, x'}_r \right)
\]

because all consumers of the same type must receive the same bundle.
Therefore, core allocations in $\mathcal{E}_r$ are just $r$-fold copies of allocations in $\mathcal{E}_1$, $\mathbf{x} = (x^1, x^2, ..., x^I)$.

*Notation*: We define the core in $\mathcal{E}_r$ as $C_r$.

We can now show that, as $r$ increases, the core shrinks.
The core shrinks as the economy enlarges:

- The sequence of core sets $C_1, C_2, \ldots$ is decreasing.
- That is, the core of the original (unreplicated) economy, $C_1$, is a superset of that in the 2-fold replica economy, $C_2$.
- In addition, the core in the 2-fold replica economy, $C_2$, is a superset of the 3-fold replica economy, $C_3$; etc.
- More compactly, $C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots \supseteq C_r \supseteq \ldots$

- Silly figure, and then proof.
Core and Equilibria

The core shrinks as the economy enlarges
The core shrinks as the economy enlarges:

Proof:

It suffices to show that, for any $r > 1$, $C_{r-1} \supseteq C_r$.

First, suppose that allocation $x = (x^1, x^2, ..., x^I) \in C_r$.

Intuitively, we cannot find any blocking coalition to $x$ in the $r$-fold replica economy $E_r$.

We now need to show that $x$ cannot be blocked by any coalition in the $(r-1)$-fold replica economy $E_{r-1}$ either.

But if could find a blocking coalition to $x$ in $E_{r-1}$ then we could also find a blocking coalition in $E_r$:

Indeed, all members in $E_{r-1}$ are also present in the larger economy $E_r$ and their endowments haven’t changed.
Core and Equilibria

- The core shrinks as the economy enlarges:
  - Graphical representation
  - See next figure.
  - In the unreplicated economy $E_1$ the set of core allocations is the line between $\tilde{x}$ and $e$
  - Some point in the line connecting $\tilde{x}$ and $e$ are WEAs and some aren’t.
    - For instance, $\tilde{x}$ is not a WEA: the price line through $\tilde{x}$ and $e$ is not tangent to $\tilde{x}$ to the consumer’s indifference curve at $\tilde{x}$.
    - In addition, note that allocation $\tilde{x}$, despite being at the core, yields the same utility level as endowment $e$ for consumer 1. That is, is the "worst" admissible allocation for consumer 1 among all core allocations.
Core and Equilibria

**In addition,** \( \bar{x} \) **yields the lowest utility for consumer 1, among all core allocations.**
Core and Equilibria

- The core shrinks as the economy enlarges:
  - Question: Does allocation $\tilde{x}$ remain at the core of the two-fold replica economy $E_2$?
  - No!
    - In particular, any point on the line connecting $\tilde{x}$ and $e$ is strictly preferred by both types of consumer 1 (he now has a twin!).
  - Let’s next try to build a blocking coalition against $\tilde{x}$:
    - We will need to guarantee:
      - Acceptance by all coalition members, and
      - Feasibility of the proposed allocation.
The core shrinks as the economy enlarges:

Building a blocking coalition against $\tilde{x}$:

Consider the midpoint allocation $\bar{x}$ and the coalition $S = \{11, 12, 21\}$.

Acceptance: If the midpoint allocation $\bar{x}$ is offered to 11 and 12, and the content in $\tilde{x}$ is offered to 21, will they accept? Yes:

\[
\overline{x}_{11} \equiv \frac{1}{2} \left( e^1 + \overline{x}_{11} \right) \succ 1 \overline{x}_{11}, \\
\overline{x}_{12} \equiv \frac{1}{2} \left( e^1 + \overline{x}_{12} \right) \succ 1 \overline{x}_{12}, \\
\overline{x}_{21} \sim 2\overline{x}_{21}
\]
The core shrinks as the economy enlarges:

Building a blocking coalition against $\tilde{x}$ (cont’d):

Feasibility: Let us now check that the suggested allocation $\{x^{11}, x^{12}, x^{21}\}$ is feasible for coalition $S$.

Since $x^{11} = x^{12}$, then the sum of the suggested allocation yields

$$\bar{x}^{11} + \bar{x}^{12} + \bar{x}^{21} = 2 \frac{1}{2} \left( e^1 + \bar{x}^{11} \right) + \bar{x}^{21}$$

$$= e^1 + \bar{x}^{11} + \bar{x}^{21}$$

Recall now that $\tilde{x}$ was part of the unreplicated economy $\mathcal{E}_1$. It then must be feasible, i.e., $\tilde{x}^1 + \tilde{x}^2 = e^1 + e^2$. Hence, $\tilde{x}^{11} + \tilde{x}^{21} = e^1 + e^2$. 
The core shrinks as the economy enlarges:

Building a blocking coalition against $\tilde{x}$ (cont’d):

Combining the above two results, we obtain

$$\bar{x}^{11} + \bar{x}^{12} + \bar{x}^{21} = e^1 + \bar{x}^{11} + \bar{x}^{21}$$

$$= e^1 + e^1 + e^2$$

$$= 2e^1 + e^2$$

Thus confirming feasibility.
Core and Equilibria

- **WEA in replicated economies.**
  - Consider a WEA in the unreplicated economy $\mathcal{E}_1$,
    $$\left(x^1, x^2, \ldots, x'\right)$$
  - Then, an allocation $x$ is a WEA for the $r$-fold replica economy $\mathcal{E}_r$ if and only if it is of the form
    $$x = \left(\underbrace{x_1, \ldots, x_1}_{r \text{ times}}, \underbrace{x_2, \ldots, x_2}_{r \text{ times}}, \ldots, \underbrace{x', \ldots, x'}_{r \text{ times}}\right)$$
  - **Proof:** If $x$ is a WEA for $\mathcal{E}_r$, then it also belongs to the core of $\mathcal{E}_r$. By the "equal treatment at the core" property, the result follows.
We are now ready to present the main result of this section.

**A limit theorem on the Core:**

- If allocation \( x \) is in the Core of the \( r \)-fold replica economy \( E_r \), for every \( r \geq 1 \), then \( x \) is a WEA for the unreplicated economy \( E_1 \).

Let’s consider, by contradiction, that an allocation \( \tilde{x} \) is not a WEA, but still belongs to the core of the \( r \)-fold replica economy \( C_r \); see next figure.

Then, \( \tilde{x} \in C_1 \) since \( C_1 \supset C_r \).

- In the next figure, this means that allocation \( \tilde{x} \) must be within the lens and on the contract curve.
Allocation $\bar{x}$ is not a WEA, but is still part of the core $e_1$.

At allocation $\bar{x}$ since $\frac{P_1}{P_2} < -MRS_{1,2}$, then $\frac{P_1}{P_2} > MRS_{1,2}$.
Core and Equilibria

- Consider the line connecting $\tilde{x}$ and $e$.
- Since $\tilde{x} \notin W(e)$, then either $\frac{p_1}{p_2} > MRS$ or $\frac{p_1}{p_2} < MRS$.
  - (The figure depicts the first case; the second is analogous.)
- By convexity of preferences, we can find a set of bundles, such as those between $A$ and $\tilde{x}$ in the figure, that consumer 1 prefers to $\tilde{x}$.
- One example of such bundle is the linear combination

$$\tilde{x} \equiv \frac{1}{r} e^1 + \frac{r-1}{r} \tilde{x}^1$$

for some $r > 1$, where $\frac{1}{r} + \frac{r-1}{r} = 1$. 
The question we now pose is, can allocation $\tilde{x}$ be at the core of the $r$-fold replica economy $E_r$ if it is not a WEA?

- No, if we can find a blocking coalition.
- Consider a coalition $S$ with all $r$ type 1 consumers and $r - 1$ type 2 consumers.
- **Acceptance:**
  - If we give every type 1 consumer the bundle $\tilde{x}^1$, we know that $\tilde{x}^1 \succ^1 \tilde{x}^1$.
  - If we give every type 2 consumer in the coalition the bundle $\tilde{x}^2 = \tilde{x}^2$, then $\tilde{x}^2 \sim^2 \tilde{x}^2$. 
Core and Equilibria

Feasibility:

Summing over the consumers in coalition $S$, their aggregate allocation is

$$r\tilde{x}^1 + (r - 1)\tilde{x}^2 = r \left[ \frac{1}{r} e^1 + \frac{r - 1}{r} \tilde{x}^1 \right] + (r - 1)\tilde{x}^2$$

$$= e^1 + (r - 1) \left( \tilde{x}^1 + \tilde{x}^2 \right)$$

Since $\tilde{x} \equiv (\tilde{x}^1, \tilde{x}^2)$ is in the core of the unreplicated economy $E_1$, then it must be feasible

$$\tilde{x}^1 + \tilde{x}^2 = e^1 + e^2$$
Core and Equilibria

- *Feasibility (cont’d):*
  - Combining the above two results, we find that

\[
\begin{align*}
  r\tilde{x}^1 + (r - 1)\tilde{x}^2 &= e^1 + (r - 1)\left( e^1 + e^2 \right) \\
  &= re^1 + r \left( e^1 + e^2 \right) - \left( e^1 + e^2 \right) \\
  &= re^1 + (r - 1)e^2
\end{align*}
\]

Thus confirming feasibility.

- Hence, *r* type 1 consumers and *r* − 1 type 2 consumers can get together in coalition *S*, and block allocation \( \tilde{x} \).
- Therefore, \( \tilde{x} \) cannot be in the Core of the *r*-fold replica economy \( \mathcal{E}_r \).
- Then, if \( \tilde{x} \in C_r \), then \( \tilde{x} \) must be a WEA.