Microeconomic Theory I - EconS 501
Final Exam - *Answer key*

1. **[Preference relations]** Consider an individual with the following preference relation in $\mathbb{R}^2_+$:

$$x \succsim y \quad \text{if and only if there is a good } k \text{ for which } x_k \geq y_k.$$

In words, the individual weakly prefers bundle x to y if and only if at least one good k is more abundant in bundle x than in bundle y.

(a) Describe the upper contour set, the lower contour set, and the indifference set.

- For a given bundle $x = (x_1, x_2)$, its upper contour set, $UCS(x)$, includes all bundles that contain more units of at least one good than bundle x does, that is,

$$UCS(x) = \left\{ y \in \mathbb{R}^2_+ : y_1 \geq x_1 \text{ or } y_2 \geq x_2 \text{ or both} \right\}.$$

As figure 1 depicts, the $UCS(x)$ includes all points weakly to the right-hand side of bundle x (areas B and D), as well as those weakly above x (areas A and B). That is, all bundles to the east, to the north, and to the northeast.

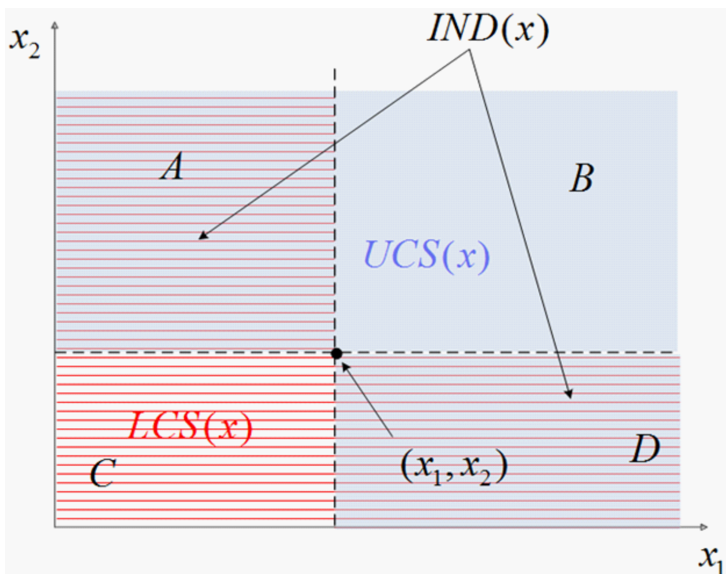

Figure 1

In contrast, the lower-contour set of bundle x is given by those bundles satisfying

$$LCS(x) = \left\{ y \in \mathbb{R}^2_+ : y_1 \leq x_1 \text{ or } y_2 \leq x_2 \text{ or both} \right\}.$$

That is, all bundles to the west, to the south, and to the southwest of bundle x, which correspond in Figure 1 to areas $A + C + D$. Graphically, this entails that the $UCS(x)$ and $LCS(x)$ overlap in all those points to the northwest of x (area A), to the southeast of x (area D), that is,

$$IND(x) = \left\{ y \in \mathbb{R}^2_+ : x_1 \leq y_1 \text{ and } x_2 \geq y_2, \text{ or } x_1 \geq y_1 \text{ and } x_2 \leq y_2 \right\}.$$

(b) Show that this preference relation is complete.

- Take two bundles $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then, we have that either:
  - the amount of good 1 is weakly larger in bundle x, $x_1 \geq y_1$, or
  - the amount of good 2 is weakly larger in bundle x, $x_2 \geq y_2$, or
  - the amounts of both goods 1 and 2 are weakly larger in bundle x.

  In all of these cases, bundle x is weakly preferred to y according the above preference relation, $x \succsim y$. As a result, the preference relation is complete.

(c) Show that this preference relation is not transitive. A counterexample suffices.

- Take three bundles $x = (1, 2)$, $y = (3, 1)$, and $z = (2, 3)$. First, note that $x \succsim y$ since bundle x contains more units of good 2 than bundle y does, $x_2 = 2 \geq 1 = y_2$. Second, $y \succsim z$ since bundle y contains more units of good 1 than bundle z does, $y_1 = 3 \geq 2 = z_1$. However, $x \not\succsim z$ since bundle z contains more units of both good 1 and 2 than bundle x does ($2 > 1$ for good 1, and $3 > 2$ for good 2). Hence, $z \succsim x$ thus violating transitivity. As a consequence, the preference relation is not rational either.

2. **[Externalities in consumption]** Consider two consumers with utility functions over two goods, x_1 and x_2, given by

$$\begin{aligned} u_A &= \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) \quad \text{for consumer } A\text{, and} \\ u_B &= \log(x_1^B) + x_2^B - \frac{1}{2}\log(x_1^A) \quad \text{for consumer } B. \end{aligned}$$

where the consumption of good 1 by individual $i = \{A, B\}$ creates a negative externality on individual $j \neq i$ (see the third term, which enters negatively on each individual's utility function). For simplicity, consider that both individuals have the same wealth, m, and that the price for both goods is 1.

(a) *Unregulated equilibrium.* Set up consumer A's utility maximization problem, and determine his demand for goods 1 and 2, as x_1^A and x_2^A. Then operate similarly to find consumer B's demand for good 1 and 2, as x_1^B and x_2^B.

- Consumer A chooses x_1^A and x_2^A to solve

$$\max_{(x_1^A, x_2^A)} \ \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B)$$

$$\text{subject to } \ x_1^A + x_2^A = M$$

  The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) + \lambda^A(M - x_1^A - x_2^A),$$

  which yields first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1^A} = \frac{1}{x_1^A} - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2^A} = 1 - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - x_1^A - x_2^A = 0$$

Solving for x_1^A, we obtain $\frac{1}{x_1^A} = 1$, i.e., $x_1^A = 1$, which implies $M - 1 - x_2^A = 0$, or $x_2^A = M - 1$. Hence, consumer A's optimal consumption is

$$x_1^A = 1 \text{ and } x_2^A = M - 1$$

A similar argument applies to consumer B,

$$x_1^B = 1 \text{ and } x_2^B = M - 1$$

(b) *Social optimum.* Calculate the socially optimal amounts of x_1^A, x_2^A, x_1^B and x_2^B, considering that the social planner maximizes a utilitarian social welfare function, namely, $W = U_A + U_B$.

- The socially optimal consumption in this case solves

$$\max_{(x_1^A, x_2^A)} U^A + U^B \quad \text{subject to } x_1^A + x_2^A = M \text{ and } x_1^B + x_2^B = M$$

The Lagrangian for this social planner's problem is

$$\mathcal{L} = \frac{1}{2}\log(x_1^A) + \frac{1}{2}\log(x_1^B) + x_2^A + x_2^B + \lambda^A(M - x_1^A - x_2^A) + \lambda^B(M - x_1^B - x_2^B)$$

Taking first-order conditions, we find the socially optimal consumption profile:

$$x_1^A = \frac{1}{2} \text{ and } x_2^A = M - \frac{1}{2}$$

$$x_1^B = \frac{1}{2} \text{ and } x_2^B = M - \frac{1}{2}$$

Intuitively, the social planner recommends a lower consumption of good 1 (the good that generates the negative externality), and an increase in the consumption of good 2, for both individuals.

(c) *Restoring efficiency.* Show that the social optimum you found in part (b) can be induced by a tax on good 1 (so the after-tax price becomes $1+t$) with the revenue returned equally to both consumers in a lump-sum transfer.[1]

- With tax t^A placed on good 1 and with lump-sum transfer T^A, consumer A solves

$$\max_{(x_1^A, x_2^A)} \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B)$$

$$\text{subject to } (1 + t^A)x_1^A + x_2^A = M + T^A$$

[1] Similarly as in the exercises about a polluting monopoly or oligopoly subject to emission fees, we assume that tax revenue is entirely returned to the agents being taxed as a lump-sum transfer. This assumption guarantees that the tax is revenue neutral, yet it helps modify agents' incentives ultimately correcting the externality, i.e., inducing the social optimum.

where note that the price of good 1 increased from 1 to $(1+t^A)$, but this consumer also sees his wealth increase by the lump sum T^A. The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) + \lambda^A(M + T^A - (1+t^A)x_1^A - x_2^A)$$

Taking first-order conditions, we obtain

$$\frac{\partial \mathcal{L}}{\partial x_1^A} = \frac{1}{x_1^A} - \lambda^A(1+t^A) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2^A} = 1 - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M + T^A - (1+t^A)x_1^A - x_2^A = 0$$

Simultaneously solving for x_1^A and x_2^A, we find that consumer A's consumption bundles after introducing the tax become

$$x_1^A = \frac{1}{1+t^A} \quad \text{and} \quad x_2^A = M + T^A - 1$$

Similarly we find the optimal consumption of consumer B who pays tax t^Bon good 1 and receives T^Bas a lump-sum transfer:

$$x_1^B = \frac{1}{1+t^B} \quad \text{and} \quad x_2^B = M + T^B - 1$$

- *Comparison.* Comparing the optimal consumption levels found in part (b) with the equilibrium outcomes found in part (c), the tax imposed on any individual $i = A, B$ must hence satisfy

$$\frac{1}{2} = \frac{1}{1+t^i},$$

which would guarantee that equilibrium and socially optimal amounts coincide. Solving for the tax t^i yields $t^i = \$1$. Hence, by setting a tax of $t^i = \$1$ on the consumption of good 1, and returning the tax revenue to this individual in a lump-sum transfer, efficiency is restored, yielding a consumption

$$x_1^i = \frac{1}{1+1} = \frac{1}{2} \text{ of good 1,}$$

and

$$\begin{aligned} x_2^i &= M + T^i - 1 \\ &= M + \frac{1}{2} - 1 = M - \frac{1}{2} \text{ of good 2,} \end{aligned}$$

as described in the socially optimal amounts found in part (b).

3. **[Subsidies to monopolists]** Assume that Alaska Airlines is a monopolist in the route Pullman-Seattle. The Washington State Legislature would like to design a policy that induces Alaska Airlines to voluntarily produce an efficient output level. Assume that the firm faces an inverse demand function $p(q)$ with $p'(q) < 0$ and marginal costs $c'(q) > 0$ for all q.

   (a) Show that such a policy must be a subsidy, and determine the exact amount of the subsidy.

   - Let t be the tax/subsidy per unit of output. Then the monopolist maximizes

     $$\max_{q \geq 0} p(q)q - c(q) - tq$$

     where, at this point, for generality we allow t to be positive (for taxes) or negative (for subsidies). Taking first order condition with respect to q yields

     $$p'(q)q + p(q) - c'(q) - t \leq 0, \text{ with equality if } q > 0$$

     Solving for t (and assuming an interior solution) we obtain

     $$t = p'(q)q + p(q) - c'(q) \tag{1}$$

     Since we seek to induce an efficient outcome, the monopolist must choose a level of output q that satisfies $p(q) = c'(q)$. This implies that the tax must be $t = p'(q)q$, so that the monopolist's profit-maximizing condition (1) becomes

     $$\underbrace{p'(q)q}_{t} = p'(q)q + p(q) - c'(q)$$

     which simplifies into the efficient outcome condition

     $$p(q) = c'(q)$$

     However, we know that $t = p'(q)q < 0$ since $p'(q) < 0$. Therefore, the tax that the government imposes on the monopoly is actually a "negative tax," i.e., a subsidy. Intuitively, the government must subsidize the monopolist in order to induce a larger level of production.

   (b) *Parametric example.* Consider that the inverse demand the firm faces is $p(q) = a - bq$ and that the cost function is $c(p) = F + cq$, where $F > 0$ denotes a fixed cost, where $a > c > 0$. Find the profit-maximizing output without subsidies, the socially optimal output, and the optimal subsidy.

   - The optimal output level q^* in this context satisfies $p(q) = c'(q)$, or $a - bq^* = c$, which yields an output level of $q^* = \frac{a-c}{b}$. In contrast, the monopolist produces $q^m = \frac{a-c}{2b}$. Therefore, the subsidy would need to be

     $$t = p'(q^*)q^* = -b\frac{a-c}{b} = c - a\,,$$

     which is a negative tax since $a > c > 0$ holds by definition. Alternatively, a subsidy of $a - c$ provides the monopolist with the incentives to produce the socially optimal output $q^* = \frac{a-c}{b}$. Indeed, if we add this subsidy into the monopolist's profit-maximization problem, we obtain

$$\max_{q \geq 0} p(q)q - c(q) + (a-c)q$$

Taking first-order conditions with respect to q, yields

$$a - 2bq^m - c + (a-c) = 0$$

which further simplifies to $a - c = bq^m$. Solving for q, we obtain a monopoly output of $q^m = \frac{a-c}{b}$, which exactly coincides with the socially optimal output $q^* = \frac{a-c}{b}$ the regulator sought to induce.

4. [**Risk aversion when you can be fired**] Consider two employees who work for the same firm. Both earn a salary of M and both consume (only) two commodities. Let x_{ik} denote agent i's consumption of commodity k. Agent 1 has preferences represented by the utility function $u_1(x_{11}, x_{12}) = \sqrt{x_{11}} + \sqrt{x_{12}}$ and agent 2's preferences are represented by $u_2(x_{21}, x_{22}) = \sqrt{x_{21}x_{22}}$. The prices of commodities 1 and 2 are p_1 and p_2.

   The firm is in the process of downsizing and one of the two employees will be laid off (and earn 0) but presently it is not known which. Each faces the probability 0.5 of being laid off.

   (a) Derive each agent's indirect utility function.

   - **Agent 1.** This individual solves the following UMP

$$\begin{aligned} \max_{x_{11}, x_{12}} \quad u_1(x_{11}, x_{12}) &= \sqrt{x_{11}} + \sqrt{x_{12}} \\ \text{subject to} \quad p_1 x_{11} + p_2 x_{12} &\leq M \end{aligned}$$

     Taking first-order conditions and solving for x_{11} and x_{12}, we obtain the Walrasian demands for good 1 and 2,

$$x_{11} = M \frac{p_2}{p_1(p_2 + p_1)} \quad \text{and} \quad x_{12} = M \frac{p_1}{p_2(p_1 + p_2)}$$

     Therefore, his indirect utility function becomes

$$v_1(p_1, p_2, M) = \sqrt{\frac{M(p_1 + p_2)}{p_1 p_2}}$$

   - **Agent 2.** This individual solves the following UMP

$$\begin{aligned} \max_{x_{21}, x_{22}} \quad u_1(x_{21}, x_{22}) &= \sqrt{x_{21}x_{22}} \\ \text{subject to} \quad p_1 x_{21} + p_2 x_{22} &\leq M \end{aligned}$$

     Taking first-order conditions and solving for x_{21} and x_{22}, we obtain the standard Walrasian demands for good 1 and 2 under a Cobb-Douglas type of utility function

$$x_{11} = \frac{M}{2p_1} \quad \text{and} \quad x_{12} = \frac{M}{2p_2}$$

Therefore, his indirect utility function becomes

$$v_2(p_1, p_2, M) = \frac{M}{2\sqrt{p_1 p_2}}$$

(b) Discuss the agents' risk postures or attitudes toward risk.

- **Agent 1.** In order to test whether agent 1 is risk averse, risk lover or risk neutral with respect to income, we need to use the indirect utility function $v_1(p_1, p_2, M)$ we found above. We are specifically interested in the concavity of $v_1(p_1, p_2, M)$. (Alternatively, you can find the Arrow-Pratt index of absolute risk aversion $r_A(M) = -\frac{\frac{\partial^2 v_1(p_1,p_2,M)}{\partial M^2}}{\frac{\partial v_1(p_1,p_2,M)}{\partial M}}$ where we find the first- and second-order derivative of $v_1(p_1, p_2, M)$ with respect to income.)[2] In particular,

$$\frac{\partial v_1(p_1, p_2, M)}{\partial M} = \sqrt{\frac{M(\frac{1}{p_1} + \frac{1}{p_2})}{2M}}$$

and

$$\frac{\partial^2 v_1(p_1, p_2, M)}{\partial M^2} = -\sqrt{\frac{M(\frac{1}{p_1} + \frac{1}{p_2})}{4M}}$$

which is negative for all parameter values (i.e., $v_1(p_1, p_2, M)$ is concave in income). From the above information the Arrow-Pratt index of absolute risk aversion is

$$r_A(M) = -\frac{\frac{\partial^2 v_1(p_1,p_2,M)}{\partial M^2}}{\frac{\partial v_1(p_1,p_2,M)}{\partial M}} = \frac{1}{2M} > 0$$

which is positive for all income levels. As a consequence, agent 1 is risk averse.

- **Agent 2.** Let us operate similarly for agent 2, using his indirect utility function $v_2(p_1, p_2, M) = \frac{M}{2\sqrt{p_1 p_2}}$.

$$\frac{\partial v_2(p_1, p_2, M)}{\partial M} = \frac{1}{2\sqrt{p_1 p_2}}$$

and

$$\frac{\partial^2 v_2(p_1, p_2, M)}{\partial M^2} = 0$$

thus implying that agent 2's indirect utility function is not concave, but linear, in income, i.e., he is risk neutral. We can confirm this point by finding the Arrow-Pratt index of absolute risk aversion is

$$r_A(M) = -\frac{\frac{\partial^2 v_2(p_1,p_2,M)}{\partial M^2}}{\frac{\partial v_2(p_1,p_2,M)}{\partial M}} = -\frac{0}{\frac{1}{2\sqrt{p_1 p_2}}} = 0$$

[2] Some of you found the derivative of $v_1(p_1, p_2, M)$ with respect to the consumption of one good. However, in order to measure risk aversion, we must evaluate the curvature of the utility function with respect to income.

(c) Compute their expected utilities.

- **Agent 1.** First, note that when this agent is laid off his income becomes zero, $M = 0$, thus yielding a zero indirect utility, i.e., $v_1(p_1, p_2, 0) = \sqrt{\frac{0(p_1+p_2)}{p_1 p_2}} = 0$. Hence, his expected utility is

$$EU_1 = \frac{1}{2}\sqrt{\frac{M(p_1+p_2)}{p_1 p_2}} + \frac{1}{2}0 = \frac{1}{2}\sqrt{\frac{M(p_1+p_2)}{p_1 p_2}}$$

- **Agent 2.** Similarly, agent 2's income when he is laid off is zero, implying a zero indirect utility $v_2(p_1, p_2, 0) = \frac{0}{2\sqrt{p_1 p_2}} = 0$. Thus, his expected utility becomes

$$EU_2 = \frac{1}{2}\frac{M}{2\sqrt{p_1 p_2}} + \frac{1}{2}0 = \frac{M}{4\sqrt{p_1 p_2}}$$

(d) Suppose they were given the opportunity to self-insure against a bad draw (earning 0) by agreeing ex ante to share whatever income they receive and each obtain $\frac{M}{2}$. Would they both necessarily accept such an agreement?

- **Agent 1.** If his income becomes $\frac{M}{2}$, this agent's utility is $\sqrt{\frac{\frac{M}{2}(p_1+p_2)}{p_1 p_2}} = \sqrt{\frac{M(p_1+p_2)}{2p_1 p_2}}$. Note that this utility exceeds his expected utility found in part (c) of $\frac{1}{2}\sqrt{\frac{M(p_1+p_2)}{p_1 p_2}}$, i.e., $\frac{1}{\sqrt{2}} \simeq 0.7$. As a result, agent 1 would accept the risk-sharing agreement.
- **Agent 2.** If his income becomes $\frac{M}{2}$, this agent's utility is $\frac{\frac{M}{2}}{2\sqrt{p_1 p_2}} = \frac{M}{4\sqrt{p_1 p_2}}$. This utility, however, exactly coincides with his expected utility found in part (c). As a result, agent 2 would be indifferent between accepting and rejecting the risk-sharing agreement.

(e) Would this risk-sharing arrangement be efficient? Discuss.

- Yes, the agreement is efficient, since the utility of one individual (agent 1) is strictly larger, while the utility of agent 2 is unaffected. In other words, the risk-sharing agreement is a Pareto improvement, since no agent is made worse off, and at least one agent is made better off.

(f) Suppose both agents believe that under the self-insurance scheme the employee who is retained by the firm would renege on the agreement and the other agent would receive nothing. Hence, they consider contracting with a third party to enforce the agreement as follows. For a fee of ε, the "enforcer" will collect an amount T from the agent who continues to work and transfer that to the other agent. In that event, both agents would bear an equal share of the fee ε. Determine an appropriate transfer T that would equalize income ex post.

- First note that the enforcer would face a setting where agent i is fired (receiving a salary of $M = 0$) while agent $j \neq i$ is still working (receiving a salary of $M > 0$). In this context, the enforcer would have to transfer T from the agent who is still working to the agent who has been fired. In order to guarantee that the transfer equalizes income ex post, we would need:

$$M - \frac{\varepsilon}{2} - T = 0 - \frac{\varepsilon}{2} + T$$

where the left-hand side of the inequality represents the income of the agent who still works (net of his share of the fee ε), while the right-hand side measures the income of the agent who was fired. Solving for T, we obtain a transfer of $T = \frac{M}{2}$.

- Hence, both agents would have an income $\frac{M}{2} - \frac{\varepsilon}{2} = \frac{M-\varepsilon}{2}$, net of their portion of the fee.

(g) What is the maximum value of ε such that both agents are willing to hire the enforcer?

- If the winning agent reneges, their expected utilities are as in part (c). With the "enforcer", their income would be $\frac{M-\varepsilon}{2}$ for both agents (as we found at the end of part f).
- **Agent 1.** With the income $\frac{M-\varepsilon}{2}$, agent 1's utility would be

$$v_1 = \sqrt{\frac{\frac{M-\varepsilon}{2}(p_1 + p_2)}{p_1 p_2}} = \sqrt{\frac{(M-\varepsilon)(p_1+p_2)}{2p_1 p_2}}$$

hence, equating $EU_1 = v_1$, we obtain

$$\frac{1}{2}\sqrt{\frac{M(p_1+p_2)}{p_1 p_2}} = \sqrt{\frac{(M-\varepsilon)(p_1+p_2)}{2p_1 p_2}}$$

Solving for ε, we can find $\varepsilon = \frac{M}{2}$, i.e., agent 1 would be willing to pay a fee up to $\frac{M}{2}$.

- **Agent 2.** Similarly, the utility of agent 2 when his income is $\frac{M-\varepsilon}{2}$ becomes

$$v_2 = \frac{\frac{M-\varepsilon}{2}}{2\sqrt{p_1 p_2}} = \frac{M-\varepsilon}{4\sqrt{p_1 p_2}}$$

Equating $EU_2 = v_2$, we find

$$\frac{M}{4\sqrt{p_1 p_2}} = \frac{M-\varepsilon}{4\sqrt{p_1 p_2}}$$

and solving for ε, we obtain $\varepsilon = 0$, i.e., agent 2 would not be willing to pay any positive fee $\varepsilon > 0$.