Exercise – Cournot competition with asymmetric costs

Consider a duopoly game in which firms compete a la Cournot, i.e., simultaneously and independently selecting their output levels. Assume that firm 1 and 2’s constant marginal costs of production differ, i.e., \( c_1 > c_2 \). Assume also that the inverse demand function is \( p(q) = a - bq \), with \( a > c_i \). Aggregate output is \( q = q_1 + q_2 \)

Find the pure strategy Nash equilibrium of this game. Under what conditions does it yield corner solutions (only one firm producing in equilibrium)?

**Answer:**

In a Nash equilibrium \((q_1^*, q_2^*)\), firm 1 maximizes its profits by selecting the output level \( q_1 \) that solves

\[
\max_{q_1 \geq 0, q_2^*} \pi_1(q_1, q_2^*) = (a - b(q_1 + q_2^*))q_1 - c_1q_1 = aq_1 - bq_1^2 - bq_2^*q_1 - c_1q_1
\]

where firm 1 takes the equilibrium output of firm 2, \( q_2^* \), as given. Similarly for firm 2, which maximizes

\[
\max_{q_2 \geq 0, q_1^*} \pi_2(q_1^*, q_2) = (a - b(q_1^* + q_2))q_2 - c_2q_2 = aq_2 - bq_1^*q_2 - bq_2^2 - c_2q_2
\]

which takes the equilibrium output of firm 1, \( q_1^* \), as given. Taking first order conditions with respect to \( q_1 \) and \( q_2 \) in the above profit-maximization problem, we obtain:

\[
\frac{\partial \pi_1(q_1, q_2^*)}{\partial q_1} = a - 2bq_1 - bq_2^* - c_1 = 0 \quad (1)
\]

\[
\frac{\partial \pi_2(q_1^*, q_2)}{\partial q_2} = a - bq_1^* - 2bq_2 - c_2 = 0 \quad (2)
\]

Solving for \( q_1 \) in (1) we obtain firm 1’s best response function:

\[
q_1(q_2) = \frac{a - bq_2 - c_1}{2b} = \frac{a - c_1}{2b} - \frac{q_2}{2b}
\]

(Note that, as usual, we rearranged the expression of the best response function to obtain two parts: the vertical intercept, \( \frac{a - c_1}{2b} \), and the slope, \( -\frac{1}{2} \)). Similarly, solving for \( q_2 \) in (2) we find firm 2’s best response function

\[
q_2(q_1) = \frac{a - bq_1 - c_2}{2b} = \frac{a - c_2}{2b} - \frac{q_1}{2b}
\]

Plugging firm 1’s best response function into that of firm 2, we obtain:

\[
q_2^* = \frac{a - c_2}{2b} - \frac{1}{2} \left( \frac{a - c_1}{2b} - \frac{q_2}{2b} \right)
\]
Since this expression only depends on $q_2$, we can now solve for $q_2$ to obtain the equilibrium output level for firm 2, $q_2^* = \frac{a + c_1 - 2c_2}{3b}$. Plugging the output we found $q_2^*$ into firm 1’s best response function $q_1 = \frac{a - 2bq_2 - c_2}{b}$, we obtain firm 1’s equilibrium output level:

$$q_1^* = \frac{a - 2b \left[ \frac{a + c_1 - 2c_2}{3b} \right] - c_2}{b} = \frac{a + c_2 - 2c_1}{3b}$$

**Corner and interior solutions:** Hence, for $q_1^* = \frac{a + c_2 - 2c_1}{3b} \leq 0$ we need firm 1’s costs to be sufficiently high, i.e., solving for $c_1$ we obtain $\frac{a + c_2}{2} \leq c_1$. A symmetric condition holds for firm 2’s output. In particular, $q_2^* = \frac{a + c_1 - 2c_2}{3b} \leq 0$ arises if $\frac{a + c_1}{2} \leq c_2$. Therefore, we can identify three different cases (two giving rise to corner solutions, and one providing an interior solution):

1. if $c_i \geq \frac{a + c_j}{2}$ for every firm $i = \{1,2\}$, then both firms’ costs are so high that no firm produces a positive output level in equilibrium;
2. if $c_i \geq \frac{a + c_j}{2}$ but $c_j < \frac{a + c_i}{2}$, then only firm $j$ produces positive output (intuitively, firm $j$ would be the most efficient company in this setting, thus leading it to be the only producer); and
3. if $c_i \leq \frac{a + c_j}{2}$ for all firm $i = \{1,2\}$, then both firms produce positive output levels.

The next figure summarizes these equilibrium results as a function of $c_1$ and $c_2$. Note that, since $a > c_i$ for every firm $i$ by assumption, we only need to look at the square region in the southwest corner of the figure (to the left of $a$ on the horizontal axis, and below $a$ on the vertical axis). In this square region, either only firm 1 produces (when $c_1$ is low and $c_2$ is high), both firms produce (when costs are relatively close to each other), or only firm 2 produces (when $c_1$ is relatively high and $c_2$ is low). In addition, the figure depicts what occurs when firms are cost symmetric, $c_1 = c_2 = c$ along the 45°-line, whereby either both firms produce positive output levels, i.e., if $c < \frac{a + c}{2}$ or $c < a$; or none of them does, i.e., if $c > a$.