Examples about Mechanism Design

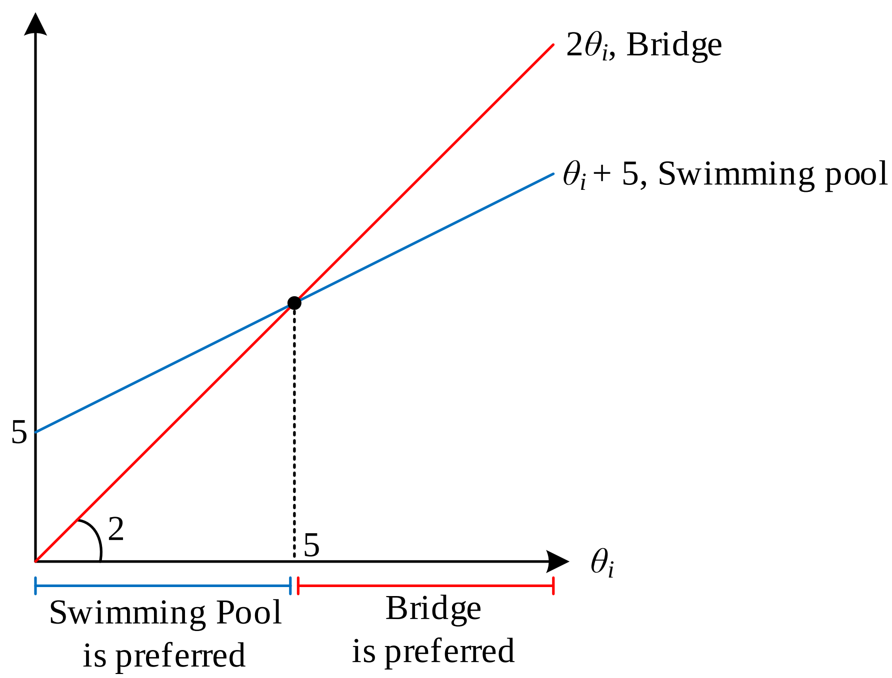
Based on Chapter 9 of JR

Example #1 (Public good project)

Consider a small town with N individuals. The town has been selected by the state to receive either a new swimming pool (S) or a new bridge (B) and must decide which it wants. Thus, the set of social states is $X = \{S, B\}$. Each individual i in the town has quasi-linear preferences and has private information θ_i regarding the value he places on the pool and on the bridge. Specifically, the values individual i places on the swimming pool (S) and on the bridge (B) are given by,

$$v_i(x, \theta_i) = \begin{cases} \theta_i + 5, & \text{if } x = S \\ 2\theta_i, & \text{if } x = B \end{cases}$$

where his type θ_i is equally likely to take on any of the values 1, 2, …, 9 and where the types are independent across individuals. The following figure depicts this valuation function.



Each individual is therefore as likely to strictly prefer the swimming pool over the bridge (i.e., $\theta_i \in \{1, 2, 3, 4\}$) as he is to strictly prefer the swimming pool over the bridge (i.e., $\theta_i \in \{6, 7, 8, 9\}$). Only the individual himself knows which of these is the case and by how much he prefers one social state over the other. In addition, the more extreme an individual's type, the more he prefers one of the social states over the other.

Example #2 (VCG mechanism in the Public Good project)

Consider the situation in Example #1. If the vector of reported types is $\theta \in \Theta$, then it is efficient for the town to build the bridge if $\sum_i v_i(B, \theta_i) > \sum_i v_i(S, \theta_i)$.[1] That is, if

$$\sum_i 2\theta_i > \sum_i \theta_i + 5$$

or, rearranging,

$$\sum_i 2\theta_i - (\theta_i + 5) = \sum_i (\theta_i - 5) > 0$$

given the definition of $v_i(.)$, this leads to the following ex-post efficient allocation function. For each $\theta \in \Theta$,

$$x^*(\theta) = \begin{cases} B, & \text{if } \sum_{i=1}^{N} (\theta_i - 5) > 0 \\ S, & \text{otherwise.} \end{cases}$$

According to the VCG mechanism, if the reported vector of types is $\theta \in \Theta$, then the social state is $x^*(\theta)$. It remains to describe the transfer, $t_i^{VCG}(\theta)$, individual i must pay. Let us think about the externality that individual i imposes on others. Suppose, for example, that the others report very high types, e.g., $\theta_j = 9$ for all $j \neq i$. Then, if there are at least two other individuals, the bridge will be built regardless of i's report. Indeed, the bridge will be built whether or not individual i is present. Hence, individual i's externality, and so also his transfer, is zero in this case. Similarly, i's externality and transfer will be zero whenever his presence does not change the outcome. With this in mind, let us define individual i as *pivotal* for social state $x \in \{S, B\}$ at the type vector $\theta \in \Theta$ when, given reports θ, his presence changes the social state from x' to x. For example, individual i is pivotal for B at $\theta \in \Theta$ if

$$\sum_{j=1}^{N} (\theta_j - 5) > 0 \text{ and } \sum_{j \neq 1}^{N} (\theta_j - 5) \leq 0,$$

[1] We assume that the swimming pool is built if the two sums are equal.

because the first (strict) inequality implies that the social state is B when he is present and the second (weak) inequality implies that it is S when he is absent. In this circumstance, i's externality and transfer is $t_i^{VCG}(\theta) = \sum_{j\neq 1}(\theta_j - 5) - \sum_{j\neq 1} 2\theta_j$, i.e., the difference between the others' total utility when he is absent and their total utility when he is present. Altogether then, $t_i^{VCG}(\theta)$ is as follows,

$$t_i^{VCG}(\theta) = \begin{cases} \sum_{j\neq 1}(5 - t_j), & \text{if } i \text{ is pivotal for } B \text{ at } \theta \in \Theta \\ \sum_{j\neq 1}(t_j - 5), & \text{if } i \text{ is pivotal for } S \text{ at } \theta \in \Theta \\ 0, & \text{otherwise.} \end{cases}$$

Example #3 (Expected externalities)

Continuing with examples #1 and #2, suppose that there are just two individuals, i.e., $N = 2$. The transfer formula given in of a VCG mechanism yields.

If your reported type, θ_i, is:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
You pay the other individual:	$\frac{10}{9}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}$	0	0	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{2}{3}$

The entries in the second row of the table are the expected VCG transfers, i.e., the $\bar{t}_i^{VCG}(\theta_i)$. In particular, the fourth entry in the second row is $\bar{t}_1^{VCG}(4)$, individual 1's expected VCG cost when he reports that his type is $\theta_1 = 4$. By reporting $\theta_1 = 4 < 5$, he can be pivotal only for the swimming pool, and even then he is pivotal only when individual 2 reports $\theta_2 = 6$, in which case his VCG cost (his externality) is $t_1^{VCG}(4, 6) = 6 - 5$ (see Example #2). Because individual 2 reports truthfully and the probability that player 2's type is $\theta_2 = 6$ is $\frac{1}{9}$, individual 1's expected externality is therefore $\bar{t}_1^{VCG}(4) = \frac{1}{9}(6 - 5) = \frac{1}{9}$, as in the table.

Note that one's payment to the other individual is higher the more extreme is one's report. This is in keeping with the idea that, for correct incentives, individuals should pay their externality (but keep in mind that the amount paid according to the table is *not* one's cost, because each individual also receives a payment from the other individual). Indeed, the more extreme an individual's report, the more likely it is that he gets his way, or, equivalently, the less likely it is that the other individual gets their way. Requiring individuals to pay more when their reports are extreme keeps them honest.

Thus, when $N = 2$, the budget-balanced expected externality mechanism for the town is as follows. The two individuals are asked to report their types and make payments to one another according to the table above. The bridge is built if the sum of the two reports exceeds 10 and the swimming pool is built otherwise. This mechanism is incentive-compatible, ex-post efficient, budget balanced, and leads to voluntary participation.

Example #4

Reconsider Example #1 but suppose that the town itself must finance the building of either the bridge or the swimming pool, and that building neither (i.e., 'Don't Build' (D)) is a third social state that is available. The types are as before as are the utilities for the bridge and pool. But we must specify utilities for building nothing. Suppose that individual 1 is the only engineer in town and that he would be the one to build the bridge or the pool. His utility for the social state D is

$$v_1(D, \theta_1) = 10,$$

while for every other individual $i > 1$,

$$v_i(D, \theta_i) = 0.$$

You may think of $v_1(D, t_1) = 10$ as the engineer's (opportunity) cost of building either the bridge or the pool. So, if the engineer cannot be forced to build (i.e., if he has property rights over the social state D), then the mechanism must give him at least and expected utility of 10 because he can ensure his utility simply by not building anything. Hence, for every profile of types $\theta \in \Theta$, we have that $u_1(\theta_1) \geq 10$ is the participation constraint (PC) of individual 1, while $u_1(\theta_i) \geq 0$ is the PC constraint of all other $i > 1$ individuals. As we now show, the expected externality mechanism that worked so beautifully without participation constraints no longer works.

Note that it is always efficient to build something, because total utility is equal to 10 if nothing is built, while it is strictly greater than 10 (assuming the engineer is not the only individual) if the swimming pool is built. Suppose that there are just two individuals, the engineer and one other. The expected externality mechanism described in Example #3 fails to work because the engineer will sometimes refuse to build. For instance, if the engineer's type is $\theta_1 < 4$, then whatever are the reports, the mechanism will indicate that either the bridge or the pool will be built and individual 2's payment to the engineer will be no more than $\frac{10}{9}$. (See the table of transfers of Example #3, where transfers are always lower or equal to $\frac{10}{9}$.) Consequently, even ignoring the payment that the engineer makes to individual 2, the engineer's expected utility if he builds is strictly less than his utility from not building 10, because

$$\max\{\theta_1 + 5, 2\theta_1\} + \frac{10}{9} < 10 \text{ when his type is } \theta_1 < 4.$$

In words, the highest utility from either the swimming pool (which yields $\theta_1 + 5$) or the bridge (which gives him $2\theta_1$) plus the highest possible transfer from individual 2 ($\frac{10}{9}$), is still lower than his utility from not building anything. For illustration purposes, the next figure illustrates max $\{\theta_1 + 5, 2\theta_1\}$, given by the upper envelope of lines $\theta_1 + 5$ and $2\theta_1$, and the parallel shift that adds the transfer $\frac{10}{9}$ to such an upper envelope. Finally, the figure also includes a flat line at 10, indicating that the engineer's utility from not building any project is higher than the most profitable project for all $\theta_1 < 4$. The engineer is, therefore, strictly better off exercising his right not to build. So, under the expected externality mechanism, the outcome is inefficient whenever $t_1 < 4$ because the engineer's participation constraint is violated.