# Chapter 15

Risk and Uncertainty

Overview

- This chapter deals with the role of risk and uncertainty in an economy. Buying stocks is an example of this scenario, where there is a potential payoff, but also a potential risk of losing money.

- We will analyze how to describe these situations, and how to predict how a decision-maker will act in the face of risk.

Lottery

- A *lottery* is simply any event in which the outcome is uncertain, much like our example about investing in stocks or the outcome of a college football game

- **Probability** is the likelihood that a specific outcome will occur (e.g. ,the likelihood of earning a positive return on the stock you purchased)

  - We can illustrate a lottery in the form of a chart…

- Furthermore, the **probability distribution** is a graph that depicts all possible outcomes and their associated probabilities, as depicted below.

Slide 1
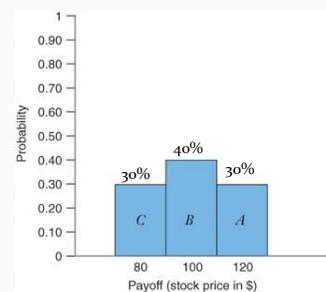
- Note that, for lotteries...
  - the probability of any particular outcome has to be between 0 and 1 $P_A \in [0,1], P_B \in [0,1], P_C \in [0,1]$
  - the sum of the probabilities of *all possible* outcomes must equal 1 $P_C + P_B + P_A = 0.3 + 0.4 + 0.3 = 1 \quad (100\%)$

- But where do probability distributions come from? That is, how do we know what the probability associated with a given event should be?

  This introduces us to *Objective* versus *Subjective Probabilities*...

Objective vs. Subjective Probabilities

- Some probabilities result from the laws of nature, as in flipping a coin. We know that there is exactly a 50% chance of flipping heads and a 50% chance of flipping tails. This is referred to as **Objective Probability.**

- However, deducing probabilities can be much harder (as in buying stocks), as there is no clear probability associated with a certain event. These are referred to as **Subjective Probabilities**: probabilities that reflect subjective beliefs about risky events.

Expected Value

- The *Expected Value* is the measure of the average payoff that the lottery will generate. That is, it is the <u>weighted average</u> of possible outcome, meaning that it takes into account the probability of each potential outcome in calculating the average.
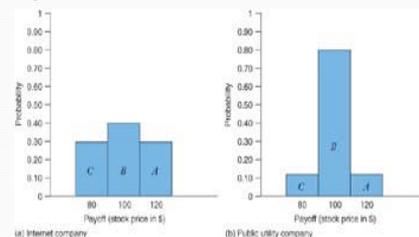
  From our book example probability distribution graphically represented with a chart.

$EV = (Prob_A \times Payoff_A) + (Prob_B \times Payoff_B) + (Prob_C \times Payoff_C)$
$EV = .30(120) + .40(100) + .30(80) = 100$

Variance

- *But*, what if some lotteries have the same EV, but one is much more volatile than the other? How do we describe this occurrence? We use a measure of variability called *variance*.

Stock of an internet company (left hand side)

$EV_{Int.} = P_A \bullet Payoff_A + P_B \bullet Payoff_B + P_C \bullet Payoff_C$

$= 0.30 \bullet \$120 + 0.40 \bullet \$100 + 0.30 \bullet \$80$

$= \$100$

Stock of a Public Utility Company (right hand side)

$EV_{P.U.} = P_A \bullet Payoff_A + P_B \bullet Payoff_B + P_C \bullet Payoff_C$

$= 0.10 \bullet \$120 + 0.80 \bullet \$100 + 0.10 \bullet \$80$

$= \$100$

Both stocks have the same EV (expected value), but...which one is less risky (lower variance)?

- **Variance** – The sum of the probability-weighted squared deviations of the possible outcomes of the lottery. That is, the variance gives us a measure of how the potential outcomes (taking into account their associated probabilities) deviate from the expected value we calculated above.

From the initial example....

Variance =

$$Prob_A \times (Payoff_A - EV)^2 + \;...(for\;each\;event)$$

Variance of the stock in the Internet Company: (Left Figure)

$Variance = P_A \bullet [Payoff_A - EV]^2 + P_B[Payoff_B - EV]^2 + P_C[Payoff_C - EV]^2$

$= 0.30[120 - 100]^2 + 0.40[100 - 100]^2 + 0.30[80 - 100]^2$

$= 0.30 \bullet 400 + 0.40 \bullet 0 + 0.30 \bullet 400$

$= 120 + 0 + 120$

$= \$240$

Variance of the Public Utility Company: (Right Figure)

$Variance = P_A \bullet [Payoff_A - EV]^2 + P_B[Payoff_B - EV]^2 + P_C[Payoff_C - EV]^2$

$= 0.10[120 - 100]^2 + 0.80[100 - 100]^2 + 0.10[80 - 100]^2$

$= 0.10 \bullet 400 + 0.80 \bullet 0 + 0.10 \bullet 400$

$= 40 + 0 + 40$

$= \$80$

Hence…

- The variance of the public utility company is smaller than that of the internet company.
- Therefore, the internet company is <u>riskier</u> since the chance of receiving a payoff different from EV is larger.

- Another common measure of riskiness (chance of payoff different than the EV) is the **standard deviation**. It is simply the square root of the variance.

$$Standard\ Deviation = \sqrt{variance}$$

<u>Example:</u>

Internet Company: std. dev.= $\sqrt{\$240} = 15.49$

Public Utility Company: std. dev.= $\sqrt{80} = 8.94$

## Utility Functions and Risk Preferences

- Say a graduate is offered two jobs upon graduation.
  - One job is from a large and well-established corporation who guarantees the graduate a salary of $54,000 for the coming year.
  - The other offer is from a new start-up company who has been operating at a loss and therefore offers the graduate a $4000 token salary. However, the company also offers the possibility of a $100,000 bonus if the company earns a profit in the ensuing year.

- What offer should/will this worker take?

- Expected Value . . .

$$EV_{Work1} = 1 \times 54000 = \$54,000$$
$$EV_{Work2} = .50 \times \$4,000 + .50 \times \$104,000 = \$54,000$$

- But, even though these two options have the same EV, the graduate will probably not look at them the same way, because of risk. We can evaluate this more formally by using a **utility function…**
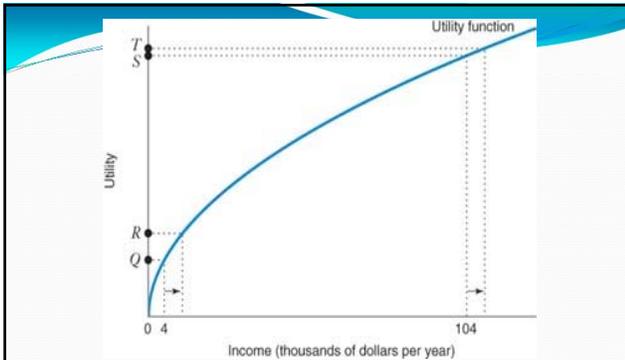
Note that this utility function satisfies the usual properties of:
- Increasing (more is better) in income
- Diminishing marginal utility- slope is decreasing.
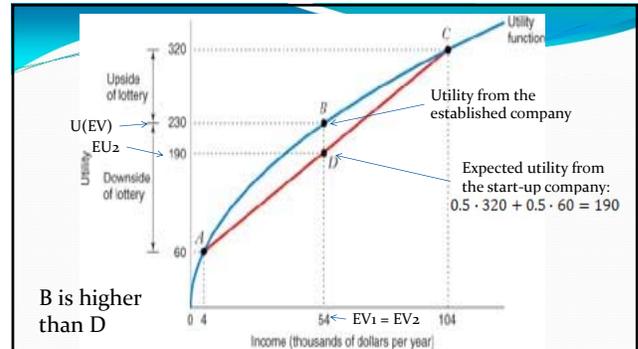
B is higher than D

- That is, even if $EV_{work\ 1} = EV_{work\ 2}$, this shows that $EU_{work\ 1} > EU_{work\ 2}$, where EU is the *Expected Utility*
- If I was offered the EV of a lottery (54) with certainty, I would prefer that (obtaining U(EV) = 230) rather than actually playing the lottery, obtaining EU = 190

- **Expected Utility** – it is simply the expected value of the utility levels that the decision maker receives from the payoffs in the lottery.

  - That is, you find the Utility from the payoff for each possible outcome of the lottery, and then find their expectation.

  - We can then use this Expected Utility to see which option the decision maker would chose.

Risk Preferences

- **Risk Averse** – A characteristic of a decision maker who prefers a sure thing to a lottery of equal expected value. That is, the decision maker above is clearly risk averse. This is represented by all concave utility functions, i.e., utility functions that increase in income, but at a decreasing rate.

- Example... $U(I) = \sqrt{100I}$
- Generally, $U(I) = A \cdot I^{\alpha}$, where $\alpha \varepsilon (0,1)$, e.g. $\frac{1}{2}$ As in
$$U(I) = A\sqrt{I}$$

Risk averse individuals

- Out of 2 lotteries (or stocks, or job offers) with the same EV, a risk averse individual selects that with the lowest variance (e.g., the job with the sure salary rather than the risky one).
- Because of diminishing marginal utility, the reduction in utility that this individual suffers from the downside of the lottery ($230-60=170) is larger than the increase in utility from the upside of the lottery ($320-230=$90).

With utility function $U(I) = \sqrt{100 \cdot I}$
- Expected Utility of Internet Company Stock:

$EU_{Internet} = .3\sqrt{100 \times 80} + .4\sqrt{100 \times 100} + .3\sqrt{100 \times 120} = 99.7$

- Expected Utility of Public Utility:

$EU_{Public} = .1\sqrt{100 \times 80} + .8\sqrt{100 \times 100} + .1\sqrt{100 \times 120} = 99.9$

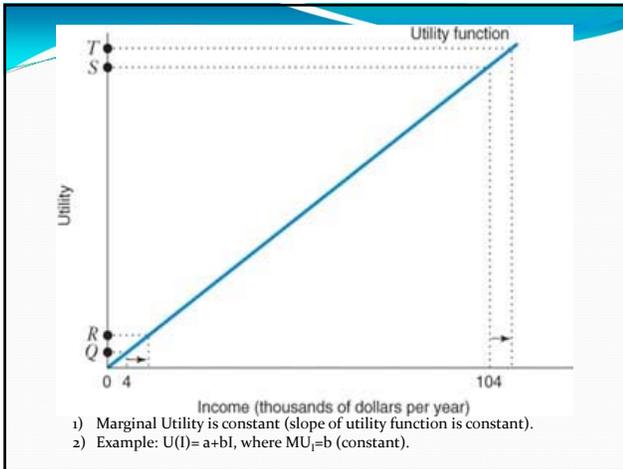- Since the $EU_{public-utility} > EU_{internetstock}$ then a risk-averse person will choose to purchase the public utility stock

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Risk averse individuals

- Hence, if two lotteries
  - Have the same EV, i.e., $EV_1 = EV_2$, but...
  - $Variance_1 > Variance_2$

- We can then anticipate that a risk averse individual will find that
  - $EU_2 > EU_1$
  - And therefore the risk averse individual prefers the lottery with the lowest variance.

- **Risk Neutral** – A characteristic of a decision maker who compares lotteries according to their expected value (not EU) and is therefore indifferent between a sure thing and a lottery with the same expected value.
  - This is characterized by a linear utility function, example: $U(I) = a + b \cdot I$, where a > 0 and b > 0

$$EU = p(a + bI_1) + (1 - p)(a + bI_2)$$

Slide 1 (top-left):

Utility function

T
S

Utility

R
Q

0 4   104

Income (thousands of dollars per year)

1) Marginal Utility is constant (slope of utility function is constant).
2) Example: U(I)= a+bI, where MU_I=b (constant).

Slide 2 (top-right):

Example

$U(I) = a + b \cdot I$, where a=0, b=100, i.e., $U(I) = 100 \cdot I$
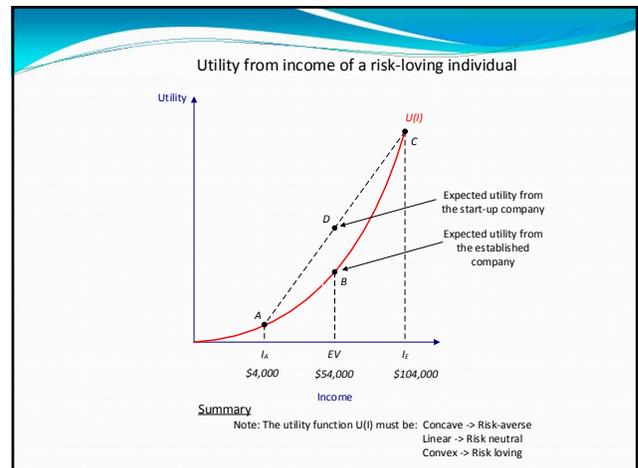
- **EU of internet stock**

  $= .3(100 \times 80) + .4(100 \times 100) + .3(100 \times 120) = 10,000$

- **EU of Public Utility**

  $= .1(100 \times 80) + .8(100 \times 100) + .1(100 \times 120) = 10,000$

  $EU_{Int}=EU_{Public\ Util}$ ➔Risk Neutral

- That is, when an individual is indifferent between two lotteries that, despite having the same EV, have different variances, we refer to this individual as "risk neutral."

Slide 3 (bottom-left):

- **Risk Loving -** A characteristic of a decision maker who prefers a lottery to a sure thing that is equal to the expected value of the lottery. In the job offer example, a risk-loving person would prefer the start-up company to the established company.

  - This is characterized by <u>increasing</u> marginal utility
  - Generally, the utility function is convex in income, as follows:

    $U(I) = A \cdot I^\beta$, where $\beta > 1$

    e.g., $\beta = 2$, A = 50 ⟶ $U(I) = 50 \cdot I^2$

  - Let's see this property in a figure

Slide 4 (bottom-right):

Utility from income of a risk-loving individual

Utility

U(I)

C

Expected utility from the start-up company

D

Expected utility from the established company

B

A

I_A EV I_E
$4,000 $54,000 $104,000

Income

Summary

Note: The utility function U(I) must be: Concave -> Risk-averse
Linear -> Risk neutral
Convex -> Risk loving

Example- Risk Lover

$U(I) = 100 \bullet I^2$

a) Exp. Utility from the Internet Stock:

$= 0.3(100 \bullet 80^2) + 0.4(100 \bullet 100^2) + 0.3(100 \bullet 120^2)$

$= 1,024,000$

b) Exp. Utility from the Public Utility Stock:

$= 0.1(100 \bullet 80^2) + 0.8(100 \bullet 100^2) + 0.1(100 \bullet 120^2)$

$= 1,008,000$

Of course, the risk lover prefers the stock with the highest variance.

- If $EV_1 = EV_2$ but $Variance_1 > Variance_2$, then a risk lover prefers lottery (or stock, or job) 1 to lottery 2.

- That is, $EU_1 > EU_2$

Bearing and Eliminating Risk

- We have described lotteries and how to calculate expected utilities to determine a consumer's preferences amongst different lotteries.

- We will now move into analyzing when an agent will choose to bear risk and when he will choose to eliminate it.

- In particular, an agent might choose to take the riskier offer if the expected payoff from the gamble is sufficiently larger than that of the sure thing.

Let's start…

- If the EV of the lottery is above the EV of the sure thing, we have

$$EU_{Lottery} > EU_{SureThing}$$

- Initially, a risk averse decision maker preferred the sure thing rather than the lottery when $EV_{Lottery} = EV_{SureThing}$.

- However, if we decrease $EV_{SureThing}$ enough, at some point the decision maker will be indifferent between the lottery and the sure thing. That is,

$$EU_{Lottery} = EU_{SureThing}$$

Definition of Risk Premium (RP):

- It is the necessary amount of money that we have to subtract from the EV in order to make the decision maker indifferent between playing the lottery and accepting the sure thing.

- That is, RP solves
  - $EU_{\text{Lottery}} = U(EV\text{-}RP)$

Figure about RP and then one numerical example...

Risk premium of the job at the start-up company
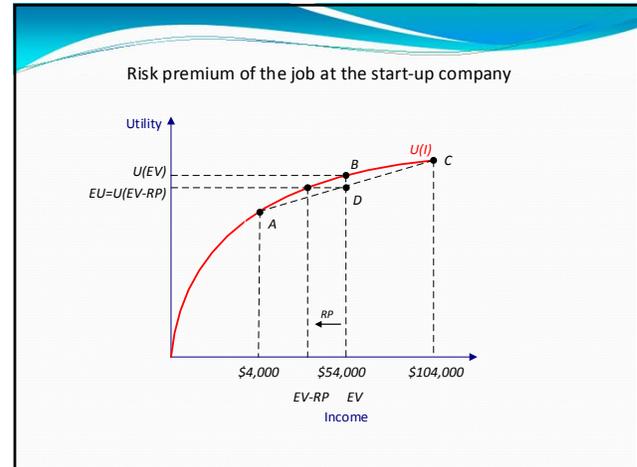
Risk Premium - Example

- A) $U = \sqrt{I}$, **Find the risk premium associated with the risky start-up company.**

- Where...
  - $I_1 = 104{,}000$
  - $I_2 = 4{,}000$
  - p=.5
  - EV=54,000

$$.5\sqrt{104{,}000} + .5\sqrt{4{,}000} = \sqrt{54{,}000 - RP}$$

$$192.87 = \sqrt{54{,}000 - RP}$$

$$37{,}199 = 54{,}000 - RP$$

$$RP = 16{,}801$$

Intuition:
This risk averse would accept a reduction of $16,801 from his sure salary of $54,000 at the established company before preferring to move to the start-up company.
e.g., A pay cut of $17,000 would induce him/her to move.

B) What if the wage offer of the start-up company changes to $I_1 = 0$, $I_2 = 108,000$ (more extreme variance!), then what is the RP?

First, note that EV is

$$EV = .5 \times 0 + .5 \times 108,000 = 54,000$$

EV is still $54,000

$$EU_{Lottery} = .5\sqrt{0} + .5\sqrt{108,000} = \sqrt{54,000 - RP}$$

$$164.32 = \sqrt{54,000 - RP}$$

- Squaring both sides,

$$27,000 = 54,000 - RP$$
$$RP = 27,000$$

- So, as the variance of the lottery increases, the RP also increases.
- Intuitively, since the worker is risk averse, he is willing to accept a larger pay cut in his sure salary of $54,000 before being induced to move to the start-up company.

$$EU_{Lottery} = U(EV-RP)$$
$$p \bullet U(I_1) + (1-p) \bullet U(I_2) = U(54,000-RP)$$

$$.5\sqrt{104,000} + .5\sqrt{4,000} = \sqrt{54,000 - RP}$$

$$192.87 = \sqrt{54,000 - RP}$$

$$37,199 = 54,000 - RP$$

$$RP = 16,801$$

<u>Intuition:</u>
This risk averse individual would accept a reduction of $16,801 from his/her sure salary of $54,000 at the established company, before preferring to move to the start-up company.
For example, a pay cut of $17,000 would induce him/her to move to the start-up company.

Fairly Priced Insurance

- The logic of risk aversion also sheds light on the circumstances under which a risk-averse person would choose to eliminate risk by buying insurance.

- *When the insurance policy has a premium equal to the expected value of the promised insurance payment, we call this* **Fairly Priced Insurance**.

⬆ What you pay every month/year

Example

<u>Example</u>: Fairly priced insurance

- Let's consider an example where a person has to decide whether or not to purchase insurance.
  - Insurance Premium = 500
  - Insurance Coverage (if accident occurs) = $10,000 (full coverage)
  - Prob of accident = .05
  - Prob of no accident = .95
    - 500 = .05(10,000) + .95(0)
      - **500=500**

    Hence, this insurance policy is fairly priced.

- **Insurance:**
  - No accident: 50,000-500 = 49,500
  - Accident: 50,000-500 -10,000 + 10,000 = 49,500
  - *So 49,500 is a sure thing, no matter what happens*

- **No Insurance:**
  - No Accident: 50,000 = 50,000
  - Accident: 50,000-10,000 = 40,000
    EV = .5(40,000) + .95(50,000) = 49,500

And so, a risk averse person will therefore pick the sure thing (buying insurance) instead of picking the lottery.

Learning-by-Doing 15.4 –
Willingness to pay for insurance

- Consider your disposable income is $90,000.
- There is a 1% chance that your house may burn, and if it does, the cost of repairing it will be $80,000, reducing your disposable income to only $10,000.
- Suppose too that your utility function is U(I)=I$^{0.5}$.

$$u(I) = \sqrt{I}$$

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a) Would you be willing to spend $500 to purchase an insurance policy that fully insures you against your loss?
  - If you don't purchase insurance, your expected utility is
    $$0.99\sqrt{90,000} + 0.01\sqrt{10,000} = \$298$$
  - If you purchase insurance at a price of $500, your disposable income becomes $89,500 whether or not your house burns.
    - Hence, your utility from insurance is
      $$\sqrt{89,500} = \$299.17$$

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a) **(Cont.)** Since your expected utility is higher with the policy than without it, you would be willing to purchase insurance at a price of $500.
b) What is the highest price that you would be willing to pay for this insurance policy?
  - Let P be the highest price that you would be willing to pay.
  - If you purchase the policy, your utility is
    $$\sqrt{90,000 - P}$$
  - Both when the house burns and when it doesn't, since the insurance policy fully compensates you for repairs

- If instead you don't purchase insurance, your expected utility is \$298, $0.99\sqrt{90,000} + 0.01\sqrt{10,000} = \$298$ (we found this in part a).
- Hence, you are indifferent between purchasing and not purchasing insurance when its price P satisfies

$$\underbrace{\sqrt{90,000 - P}}_{\sqrt{EV - RP}} = \underbrace{298}_{EU_{Lottery}} \rightarrow 90,000 - P = 88,804$$

- That is, the most you'd be willing to pay for this insurance policy is \$1,196.

Asymmetric Information

- **Asymmetric Information** – *it is basically a situation in which one party knows more about its own actions or characteristics than another party*.

- For instance, you know more about your driving habits than a car insurance company.
- For this reason, car insurance companies charge a premium to safeguard against their lack of information
- Similarly, think about health insurance deductibles.

Suppose your car is fully insured. How careful would you be? Probably not as careful as you would be without insurance, or with partial insurance. That is, insurance in a way incentivizes less careful driving.

- This is a form of **moral hazard** – a phenomenon whereby an insured party exercises less care than he or she would in the absence of insurance.
- How to provide incentives for careful driving?
  - Including deductible into the insurance policy.
  - But not too big! Otherwise good drivers (if very risk averse) might not be attracted to the policy.

- Another reason insurance companies do not provide full coverage is **adverse selection** – a phenomenon whereby an increase in the premium increases the overall riskiness of the pool of individuals who buy an insurance policy.

  - That is, the higher the premium the insurance company charges, the more likely that only people truly in need of insurance (reckless car drivers, unhealthy people, etc.) will buy the policy

## How to avoid adverse selection?

- One solution: Offering a "menu" of insurance policies to customers, and allow each customer to select the particular policy that he/she most prefers.
  - A policy with a **large** deductible and **low** premium would appeal to someone who is convinced his/her chances of illness are **low**, whereas...
  - A policy with a **small** deductible and **high** premium would appeal to someone who is convinced that his/her chances of illness are **high**.

How to avoid adverse selection?

- *Another solution*:
  - **Pooling good and bad risks together:**
  - For instance, if all employees in a company participate in a mandatory companywide health insurance plan, the insurance company offering the group plan will face a mix of high and low-risk individuals, reducing its costs as a consequence.
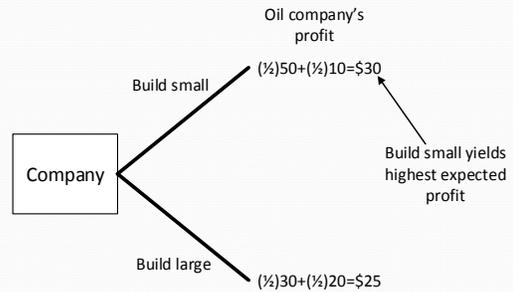
Decision trees

- A *Decision tree* is a diagram that describes the options available to a decision maker...
  - as well as the risky events that can occur at each point in time.

- Let's see one simple example for an oil company that has just discovered a new reserve of oil
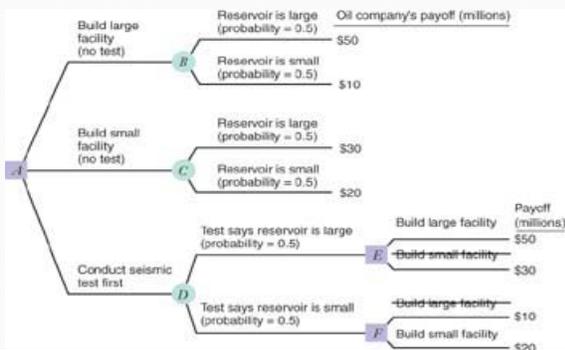 - (next page).
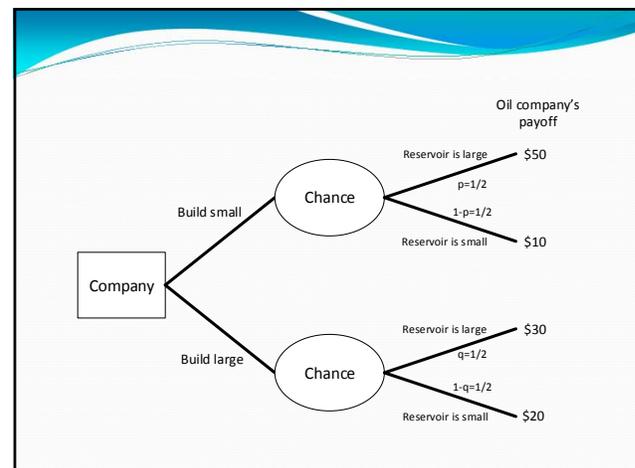
Decision trees – Main elements

- *Decision node*: they indicate a particular decision that the decision maker faces.
  - Each branch from a decision node corresponds to a possible decision.
- *Chance nodes*: They indicate a particular lottery that the decision maker faces.
  - Each branch from a chance node corresponds to a possible outcome of the lottery.
- *Probabilities*: Each possible outcome has a probability.
- *Payoffs*: Each branch at the right-hand end of the tree has a payoff associated to it.
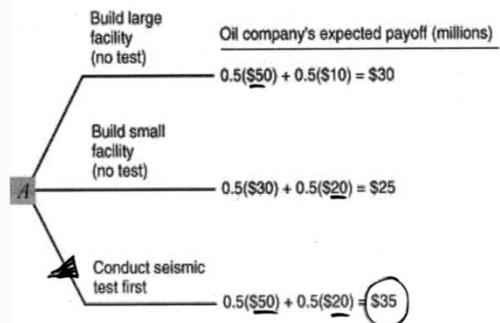 - The payoff is the value of the result from each possible combination of choices and risky outcomes.

Folded back decision tree for oil company's facility size decision:

Oil company's profit

Build small — (½)50+(½)10=$30

Build small yields highest expected profit

Company

Build large — (½)30+(½)20=$25

An enlarged decision tree where the oil company can conduct a seismic test (chance node D)

Folded back decision tree…

Build large facility (no test) — Oil company's expected payoff (millions)
0.5($50) + 0.5($10) = $30

Build small facility (no test) — 0.5($30) + 0.5($20) = $25

Conduct seismic test first — 0.5($50) + 0.5($20) = $35

Auctions

- Auctions are a large part of the economic landscape (i.e. governments auctioning off their air waves, eBay, etc.).
- Auctions typically involve relatively few decision makers who make decisions under uncertainty.

- **Private value auctions:** Each buyer has his own personal valuation of the auctioned item.
  - You know the value you assign to the object for sale, but not the value other bidders assign to the object
  - *Examples:* Art, antiques.

- **Common value auctions:** The item being auctioned has the same intrinsic value to all buyers, but no buyer knows exactly what that value is.
  - *Examples:* Oil leases, Government treasury bills.

- **Types of Private Value Auctions**

  - *English Auction* – an auction in which participants cry out their bids and each participant can increase his or her bid until the auction ends with the highest bidder winning the object being sold.

  - *First-Price Sealed Bid Auctions* – an auction in which each bidder submits one bid, not knowing the other bids. The highest bidder wins the object and pays a price equal to his or her bid.

- *Second-Priced Sealed-Bid Auction* – An auction in which each bidder submits one bid, not knowing the other bids. The highest bidder wins the object but pays an amount equal to the second-highest bid.
  - Airwave licenses in New Zealand

- *Dutch Descending Auction* – An auction in which the seller of the object announces a price which is then lowered until a buyer announces a desire to buy the item at that price.
  - Often used for agricultural commodities (tobacco, tulips in Holland, fish in JPN.)

Auctions

- *N* bidders, each bidder *i* with a valuation v_i for the object.
- One seller.
- We can design many different rules for the auction:
  1. *First price auction*: the winner is the bidder submitting the highest bid, and he/she must pay the <u>highest bid</u> (which is his/hers).
  2. *Second price auction*: the winner is the bidder submitting the highest bid, but he/she must pay the <u>second highest bid</u>.
  3. *Third price auction*: the winner is the bidder submitting the highest bid, but he/she must pay the <u>third highest bid</u>.
  4. *All-pay auction*: the winner is the bidder submitting the highest bid, but <u>every single bidder</u> must pay the price he/she submitted.

Auctions

- Note that all auctions can be interpreted as allocation mechanisms: each auction specifies
  1. **An allocation rule** (who gets the object):
     1. The allocation rule for most auctions determines the object is allocated to the individual submitting the highest bid. However, we could assign the object by a lottery, where $prob(win) = \frac{b_1}{b_1 + b_2 + \cdots + b_N}$ as in "Chinese auctions".
  2. **A payment rule** (how much every bidder must pay):
     1. The payment rule in the FPA determines that the individual submitting the highest bid pays his/her bid, while everybody else pays zero.
     2. The payment rule in the SPA determines that the individual submitting the highest bid pays the second highest bid, while everybody else pays zero.
     3. The payment rule in the APA determines that every individual must pay the bid he/she submitted.
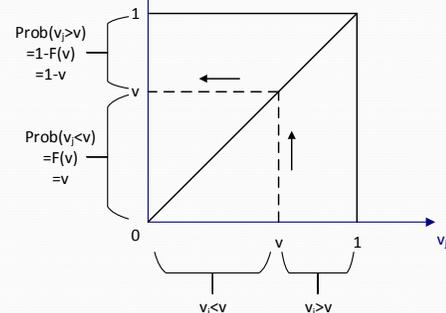
Auctions

- Auctions as incomplete information games:
  1. I know my valuation for the object, v_i, but...
  2. I don't know your valuation for the object, v_j, but I know that it is drawn from a distribution function.
     1. Easiest case:
        $$v_j = \begin{cases} 10 \text{ with probability } 0.4, \text{ or} \\ 5 \text{ with probability } 0.6 \end{cases}$$
     2. More generally,
        c.d.f. $\rightarrow$ F(v) = prob(v_j<v)
     3. We will normally assume that every bidder's valuation for the object is drawn from a uniform distribution function between 0 and 1, [0,1].
  3. Symmetry in the bidding function:
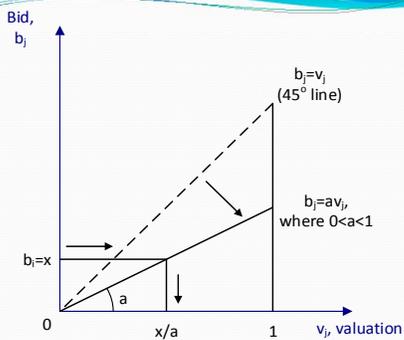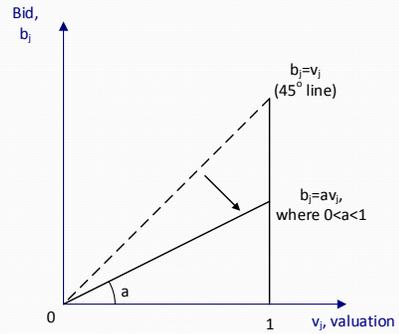 $$b_i : [0,1] \rightarrow \mathbb{R}_+ \quad \text{for every bidder i}$$

## First Price Auctions

- Never bid above your value, $b_i > v_i$
  - Negative Payoff if winning
    $$EU_i = prob(win) \bullet (v_i - b_i) + prob(lose) \bullet 0$$
- Never bid your value, $b_i = v$
  - Zero payoff if winning,
    $$EU_i = prob(win) \bullet (v_i - b_i) + prob(lose) \bullet 0$$

- Then bidders "shade their valuation" by bidding below their value, $b_i < v_i$.
  - In particular,
    $$b_i = a \bullet v_i, \text{ where } 0 < a < 1$$

"Bid Shading" in the FPA

A bid of $b_i = x$ is made by a bidder with valuation v_i, satisfying:
$$x = b_i = av_i \rightarrow x = av_i \rightarrow x/a = v_i$$
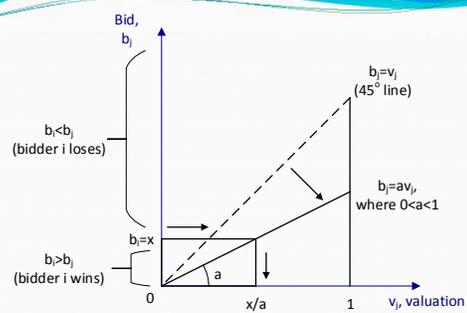
Valuations of bidder j, v_j, for which $b_i > b_j$

Valuations of bidder j, v_j, for which $b_i < b_j$

What is the probability of winning?

- Of course, prob(win)= prob($b_i > b_j$)
- And, according to the previous figure,

  prob($b_i > b_j$)=prob($x > b_j$)
- And from the point of view of valuations (horizontal axis), prob($x > b_j$)=prob($\frac{x}{a} > v_i$)

See shaded region of vertical axis

See shaded region of horizontal axis

- Assuming that all valuations are uniformly distributed between 0 and 1, prob($\frac{x}{a} > v_j$)=$\frac{x}{a}$
- Recall that:

  Prob($x > v_j$)=x

  Prob($x < v_j$)=1-x
- We are now ready to write the bidder's expected utility b_i=x, for a given valuation v_i,

$$EU_i(b_i|v_i)=(v_i-x)\bullet\frac{x}{a}+0\bullet(1-\frac{x}{a})$$

Payoff if winning   Prob of winning   Payoff if losing   Prob of losing

$$=\frac{v_i x - x^2}{a}$$

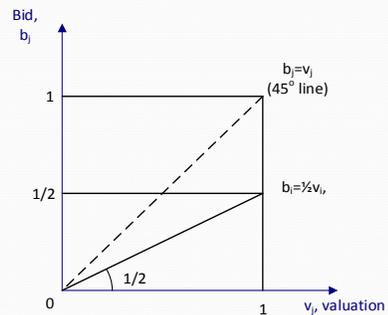Taking F.O.C. with respect to bidder *i*'s bid, x,

$$\frac{v_i - 2x}{a} = 0 \rightarrow v_i = 2x$$

And solving for the bid *x*, we obtain the optimal bidding function

$$x(v_i) = \frac{1}{2}v_i$$

That is, the bidder submits bid equal to half of his valuation for the object.

Optimal bidding function $x(v_i) = \frac{1}{2}v_i$

Bid, b_j

b_j=v_j (45° line)

1

1/2

b_i=½v_i,

1/2

0

1

v_j, valuation

What if we are dealing with *N* bidders?

$\text{Prob(win)}=\text{prob}(\frac{x}{a}>v_1)\bullet\text{prob}(\frac{x}{a}>v_2)...\bullet\text{prob}(\frac{x}{a}>v_{i-1})$
$\bullet\text{prob}(\frac{x}{a}>v_{i+1})...\bullet\text{prob}(\frac{x}{a}>v_N)$

$=\frac{x}{a}\bullet\frac{x}{a}...\bullet\frac{x}{a}=(\frac{x}{a})^{n-1}$

Therefore, the EU_i is:

$$EU_i(b_i|v_i)=(v_i-x)\bullet(\frac{x}{a})^{n-1}+0\bullet[1-(\frac{x}{a})^{n-1}]$$

| Payoff if winning | Prob of winning | Payoff if losing | Prob of losing |

Taking F.O.C. with respect to bid *x*:

$$-(\frac{x}{a})^{n-1}+(v_i-x)\bullet(n-1)\bullet(\frac{x}{a})^{n-2}\bullet\frac{1}{a}=0$$

Rearranging,

$$\frac{1}{x}\bullet a\bullet(\frac{x}{a})^{n}\bullet[(n-1)v_i-nx]=0$$

And solving for *x*,

$$(n-1)v_i=nx$$

$$\boxed{x(v_i)=\frac{n-1}{n}\bullet v_i}$$

Optimal bidding function

Note that for very large N, e.g., N=2,000, we have $b_i=\frac{1,999}{2,000}v_i$, and $b_i\cong v_i$ (almost coinciding with the bidder's valuation, in 45° line, i.e., very small "bid shading").
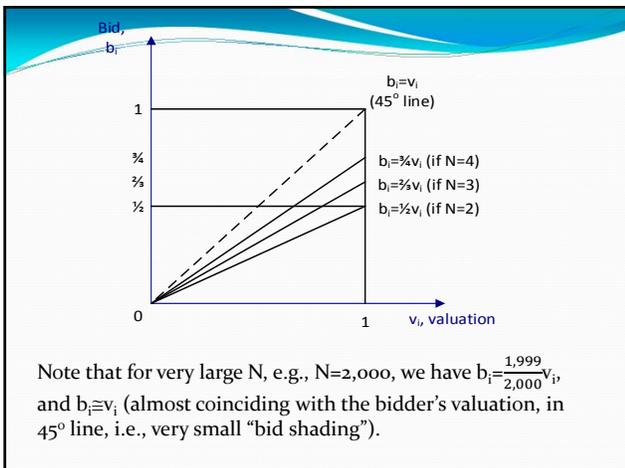

Second Price Auction, SPA

- Bidding your valuation, $b_i=v_i$, is a <u>weakly dominant</u> strategy for all players.
  - That is, there is no other bidding strategy that provides a bidder with a strictly larger payoff.
  - In other words, all other strategies yield a lower or equal payoff as bidding my own valuation, $b_i=v_i$.

Second Price Auction, SPA

- Let's show it by comparing the payoffs from bidding $b_i=v_i$ versus any other bidding strategy.
  1. If the bidder bids $b_i=v_i$, then either
     a) $h_i>b_i$, and he loses the auction, or
     b) $h_i<b_i$, and he wins with a payoff v_i-h_i, or
     c) $h_i=b_i$, a tie, then the object is randomly assigned
        $\frac{1}{2}(v_i-h_i)$
        (Or among the number of K bidders submitting the same highest bid)

(-) Payoff $\quad v_i \quad$ (+) Payoff
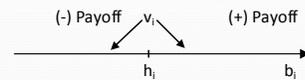
$h_i \qquad b_i$

2. If the bidder shades his bid, $b_i<v_i$, then either
   a) $h_i>b_i$, and he loses the auction, or
   b) $h_i<b_i$, and he wins with a payoff v_i-h_i, or
   c) $h_i=b_i$, a tie occurs, and the object is randomly assigned yielding a expected payoff of $\frac{1}{2}(v_i-h_i)$

(-) Payoff $\quad v_i \quad$ (+) Payoff
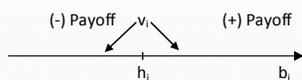
$h_i \qquad b_i$

2. If the bidder bids above his valuation, $b_i>v_i$, either
   a) $h_i>b_i$, and he loses the auction, or
   b) $h_i<b_i$, and he wins, and:
      He gets a payoff of $v_i-h_i>0$ if $v_i>h_i$, or
      He gets a negative payoff if $v_i<h_i$.
   c) $h_i=b_i$, a tie, with the expected payoff:
      $\frac{1}{2}(v_i-h_i)>0$ if $v_i>h_i$
 $\frac{1}{2}(v_i-h_i)<0$ if $v_i<h_i$

Second-price auctions

- **Summary:**
- Hence, there is no bidding strategy providing a strictly higher payoff than $b_i=v_i$ in the SPA.
- Efficiency:
  1. In auctions, we say that an auction (or any allocation mechanism) is efficient if the bidder with the <u>highest valuation for the object</u> is indeed the person receiving the object.
  2. In this sense, both the FPA and the SPA are efficient.

English auction

- Easy!
- They are equivalent to second-price auctions (SPA) since...
  - If you are the bidder with the highest willingness to pay for the good, you increase your bid until the last bidder drops and...
  - You pay just a few bit more than the second highest bid (the last bid your closest competitor submitted).

- **<u>Common Value Auctions:</u>**
  - When bidders have common values, a complication arises that does not occur when bidders have private values, the **winner's curse**: The winning bidder might bid an amount that exceeds the item's intrinsic value

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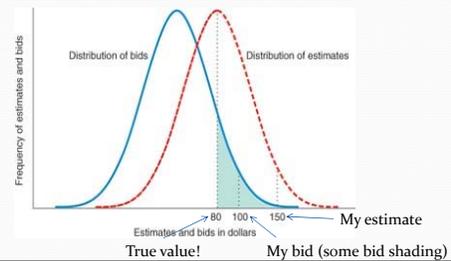
Winner's curse

- In auctions where all bidders assign the same valuation to the object (common value auctions), and where every bidder receives an inexact signal of the object's true value...
- The fact that you won just means that you received an overestimated signal of the true value of the object for sale (oil lease).
- How to avoid the winner's curse?
  - Bid b=s-2 or less, in order to take into account the possibility that you receive an overestimated signal.

Winner's curse

- Empirically tested:
  - *The winner's curse in the classroom:*
    - A jar of nickels which every student can look at for a few minutes before submitting his/her bid

Winner's curse

- Empirically tested:
  - *The winner's curse in the field:*
    - Texaco in auctions selling mineral rights to off-shore properties owned by the US government.
    - All firms avoided the winner's curse, except for Texaco.
    - Were the executives sent back to school for some remedial microeconomics?

More on auction theory and related topics on…

Course on
Strategy and Game Theory
EconS 424 – Spring 2015