Optimal bidding function in Common Value Auctions

In previous sections we showed that in common value auctions every bidder must shade his bid (i.e., submit a bid lower than his own valuation) as otherwise he could fall prey of the "winner's curse", that is, winning the auction but paying for the good a price higher than his valuation. However, we were silent about how much bid shading is optimal in equilibrium. The following discussion, based on Harrington's textbook, analyzes how to find optimal bidding functions in common-value auctions.

Suppose a common value auction with $n \geq 2$ bidders. The true value of the object being auctioned is v and is the same for all bidders. Each bidder gets a noisy (or inexact) signal of v. For simplicity, assume that such a signal is drawn from the interval $[0,1]$ according to a uniform distribution. The cumulative distribution function on bidder i's signal, denoted s_i, is

$$F(s_i) = \begin{cases} 0 \; if \; s_i < 0 \\ s_i \;\; if \; 0 \leq s_i \leq 1. \\ 1 \; if \; 1 < s_i \end{cases}$$

The signal of bidder i is known only to him; thus, a bidder's signal is his type and the type space is $[0,1]$. It is common knowledge that each bidder's signal is independently drawn from $[0,1]$ according to F. Finally, it is assumed that the true value is randomly determined by Nature in that it is assumed to equal the average of all bidders' signals:

$$v = \left(\frac{1}{n}\right) \sum_{j=1}^{n} s_j. \qquad (1)$$

Bidders participate in a first-price, sealed-bid auction, which means that if bidder i wins, then his realized payoff is $v - b_i$ where b_i is his bid, though he doesn't learn v until after he has won.

In deriving a BNE, let us conjecture that it is linear in a bidder's signal. That is, there is some value for $\alpha > 0$ such that

$$b_j = \alpha s_j. \qquad (2)$$

where α represents bid shading.

Constructing expected utility. Bidder i's expected payoff is the probability that he wins (i.e., his bid is higher than all other bids) times his expected payoff, conditional on hiving submitted the highest bid:

$$Prob\left(b_i > b_j \; for \; all \; j \neq i\right) \times \{E\left[v \middle| s_i, b_i > b_j \; for \; all \; j \neq i\right] - b_i\} \qquad (3)$$

$E[v|s_i, b_i > b_j$ for all $j \neq i]$ is bidder i's expected valuation, conditional not only on his signal, but also on knowing that he submitted the highest bid. This latter fact says something about the signals of the other bidders and thus about the true value of the object.

Now let us use the property that other bidders are conjectured to use the bidding rule in (2). Substitute αs_j for b_i in (3):

$$\text{Prob}(b_i > \alpha s_j \text{ for all } j \neq i) \times \{E[v|s_i, b_i > \alpha s_j \text{ for all } j \neq i] - b_i\}$$
$$=\text{Prob}\left(\frac{b_i}{\alpha} > s_j \text{ for all } j \neq i\right) \times \{E[v|s_i, \frac{b_i}{\alpha} > s_j \text{ for all } j \neq i] - b_i\}.$$

Next, substitute the expression for v from (1):

$$\text{Prob}(\frac{b_i}{\alpha} > s_j \text{ for all } j \neq i) \times \{E[\left(\frac{1}{n}\right)(s_i + \textstyle\sum_{j\neq i} s_j)|\, \frac{b_i}{\alpha} > s_j \text{ for all } j \neq i] - b_i\}$$

Since bidder i knows s_i, so that $E[s_i] = s_i$, but does not know s_j, we can rearrange this expression as

$$\text{Prob}(\frac{b_i}{\alpha} > s_j \text{ for all } j \neq i) \times \{\frac{s_i}{n} + \frac{1}{n} E[\textstyle\sum_{j\neq i} s_j|\, \frac{b_i}{\alpha} > s_j \text{ for all } j \neq i] - b_i\}$$

And because signals are independent random variables, we can move the sum operator outside the expectation operator,

$$\text{Prob}(\frac{b_i}{\alpha} > s_j \text{ for all } j \neq i) \times \{\frac{s_i}{n} + \frac{1}{n}\textstyle\sum_{j\neq i} E\left[s_j \middle| \frac{b_i}{\alpha} > s_j\right] - b_i\}.$$

Using the uniform distribution on s_j, we see that bidder i's expected payoff becomes

$$\left(\frac{b_i}{\alpha}\right)^{n-1} \times \left[\frac{s_i}{n} + \frac{1}{n}\sum_{j\neq i}\frac{b_i}{2\alpha} - b_i\right]$$

where $\text{Prob}(\frac{b_i}{\alpha} > s_j \text{ for all } j \neq i) = \left(\frac{b_i}{\alpha}\right)^{n-1}$, and $E\left[s_j | \frac{b_i}{\alpha} > s_j\right] = \frac{\frac{b_i}{\alpha} - 0}{2} = \frac{b_i}{2\alpha}$ since we find the expectation of s_j for all values between 0 and $\frac{b_i}{\alpha}$. Summing over all $j \neq i$ (with $N - 1$ components), we obtain

$$\left(\frac{b_i}{\alpha}\right)^{n-1} \times \left[\frac{s_i}{n} + \frac{n-1}{n}\frac{b_i}{2\alpha} - b_i\right] \tag{4}$$

Note that the last expression presumes that $\frac{b_j}{\alpha} \leq 1$. Since we will show that $b_i = \alpha s_i$ for some value of α, it follows that $\frac{b_j}{\alpha} \leq 1$ is equivalent to $\frac{\alpha s_i}{\alpha} \leq 1$, or $s_i \leq 1$, which is true by assumption.

Ready to take FOC!! Bidder i chooses b_i to maximize (4). The first-order condition with respect to b_i is

$$(n-1)\left(\frac{1}{\alpha}\right)\left(\frac{b_i}{\alpha}\right)^{n-2}\left[\left(\frac{s_i}{n}\right) + \left(\frac{n-1}{n}\right)\left(\frac{b_i}{2\alpha}\right) - b_i\right] + \left(\frac{b_i}{\alpha}\right)^{n-1}\left(\frac{n-1-2\alpha n}{2\alpha n}\right) = 0.$$

Solving this equation for b_i we obtain

$$b_i = \left(\frac{2\alpha}{2\alpha n-(n-1)}\right)\left(\frac{n-1}{n}\right) s_j. \tag{5}$$

Recall that we conjectured that the symmetric equilibrium bidding rule is $b_i = \alpha s_i$ for some value of α by equating to the coefficient multiplying s_i in (5):

$$\alpha = \left(\frac{2\alpha}{2\alpha n - (n-1)}\right)\left(\frac{n-1}{n}\right)$$

Solving this equation for α, we get

$$\alpha = \left(\frac{(n+2)(n-1)}{2n^2}\right)$$

In conclusion, a symmetric BNE has a bidder using the rule

$$b_i = \left(\frac{n+2}{2n}\right)\left(\frac{n-1}{n}\right)s_i.$$

Comparative statics. We can finally check that, when the number of bidders is only $n = 2$, the optimal bidding function becomes $b_i = \frac{1}{2}s_i$. When the number of bidders increases to $n = 3$, the optimal bidding function rotates upward to $b_i = \frac{5}{9}s_i$. However, when the number of bidders further increases, for instance to $n = 10$, the optimal bidding function now rotates downard to $b_i = \frac{27}{50}s_i$. More generally, the derivative of the optimal bidding function with respect to n is

$$\frac{\partial b_i}{\partial n} = \frac{(4-n)}{2n^3}s_i$$

which is positive (i.e., more aggressive bidding as n grows) when $n < 4$, but negative (i.e., less aggressive bidding as n grows) otherwise.